MCINTYRE MATHS · YEAR 10 MATHEMATICS

Worked solutions

Lesson 5 · Factorising polynomials to find zeros

Warm-up · Factors and the factor theorem

(a) Solve $(x-3)(x+5)=0$.

> **$x=3$ or $x=-5$**
> A product is zero when one factor is zero. Solve $x-3=0$ and $x+5=0$.

(b) Factorise x^2+x-12.

> **$(x+4)(x-3)$**
> We need two numbers that multiply to -12 and add to 1: 4 and -3.

(c) Let $P(x)=x^3-2x^2-5x+6$. Find $P(1)$. What does your answer tell you about $(x-1)$?

> **$P(1)=0$, so $(x-1)$ is a factor.**
> $1-2-5+6=0$. By the factor theorem, $P(c)=0$ means $(x-c)$ is a factor.

Example 1 · Zeros from factors

Find the zeros of the polynomial $P(x)=(x+2)(2x+1)(x-4)$.

> **The zeros are -2, $-\frac{1}{2}$ and 4.**
> Solve $P(x)=0$. Equate each factor to zero: $x+2=0$ gives $x=-2$, $2x+1=0$ gives $x=-\frac{1}{2}$, and $x-4=0$ gives $x=4$. Solve $2x+1=0$ fully: the answer is not $\frac{1}{2}$.

Your turn · Null factor law

(a) $(x-1)(x+6)(x-2)=0$

> **$x=1$, -6 or 2**
> Equate each factor to zero.

(b) $(3x-2)(x+4)(2x+5)=0$

> **$x=\frac{2}{3}$, -4 or $-\frac{5}{2}$**
> $3x-2=0$ gives $x=\frac{2}{3}$, and $2x+5=0$ gives $x=-\frac{5}{2}$.

(c) $x(x-5)(2x+1)=0$

> **$x=0$, 5 or $-\frac{1}{2}$**
> The factor x gives $x=0$. Do not leave it out.

(d) $(x+1)^2(x-3)=0$

> $x=-1$ **or** 3
> The factor $(x+1)$ is repeated, so $x=-1$ is listed once. It is a double root: the graph touches the x-axis there instead of crossing it.

Example 2 · Using the constant term

Solve $x^3+3x^2-6x-8=0$.

> $x=-1, -4$ **or** 2
> Let $P(x)=x^3+3x^2-6x-8$. The constant term is -8, so the possible values of c are $\pm1, \pm2, \pm4, \pm8$. $P(-1)=-1+3+6-8=0$, so $(x+1)$ is a factor.

$$
\begin{array}{rr|rrr}
 & & x^2 & +2x & -8 \\
\hline
x+1 & x^3 & +3x^2 & -6x & -8 \\
- & x^3 & +x^2 & & \\
\hline
 & & 2x^2 & -6x & \\
 & - & 2x^2 & +2x & \\
\hline
 & & & -8x & -8 \\
 & - & & -8x & -8 \\
\hline
 & & & & 0
\end{array}
$$

> The quotient is $x^2+2x-8=(x+4)(x-2)$. So $P(x)=(x+1)(x+4)(x-2)$ and the solutions are $x=-1, -4$ and 2. Remember the factor is $(x-c)$, so $P(-1)=0$ gives $(x+1)$.

Your turn · Using the constant term

Solve $x^3-4x^2+x+6=0$.

> $x=-1, 2$ **or** 3
> The constant term is 6. $P(1)=1-4+1+6=4$ is not zero, so keep testing. $P(-1)=-1-4-1+6=0$, so $(x+1)$ is a factor.

$$
\begin{array}{rr|rrr}
 & & x^2 & -5x & +6 \\
\hline
x+1 & x^3 & -4x^2 & +x & +6 \\
- & x^3 & +x^2 & & \\
\hline
 & & -5x^2 & +x & \\
 & - & -5x^2 & -5x & \\
\hline
 & & & 6x & +6 \\
 & - & & 6x & +6 \\
\hline
 & & & & 0
\end{array}
$$

> The quotient is $x^2-5x+6=(x-2)(x-3)$. So $(x+1)(x-2)(x-3)=0$ and $x=-1, 2$ or 3 .

Example 3 · Only one solution

Show that $x^3 - 3x^2 + 4x - 2 = 0$ has exactly one real solution.

The only real solution is $x = 1$.
Let $P(x) = x^3 - 3x^2 + 4x - 2$. $P(1) = 1 - 3 + 4 - 2 = 0$, so $(x - 1)$ is a factor.

$$
\begin{array}{rrrrr}
 & & x^2 & -2x & +2 \\
\cline{2-5}
x-1 & \multicolumn{1}{|r}{x^3} & -3x^2 & +4x & -2 \\
- & x^3 & -x^2 & & \\
\cline{2-3}
 & & -2x^2 & +4x & \\
- & & -2x^2 & +2x & \\
\cline{3-4}
 & & & 2x & -2 \\
- & & & 2x & -2 \\
\cline{4-5}
 & & & & 0
\end{array}
$$

$P(x) = (x - 1)(x^2 - 2x + 2)$. For $x^2 - 2x + 2$ the discriminant is $(-2)^2 - 4(1)(2) = -4$, which is negative, so it has no real zeros. A negative discriminant means no real solutions. The only real solution is $x = 1$.

Example 4 · A common factor

Solve $x^3 - 2x^2 - 6x = 0$, giving exact answers.

$x = 0$, $x = 1 + \sqrt{7}$ or $x = 1 - \sqrt{7}$
There is no constant term, so x is a common factor: $x(x^2 - 2x - 6) = 0$. So $x = 0$ or $x^2 - 2x - 6 = 0$. By the quadratic formula, $x = \dfrac{2 \pm \sqrt{4 + 24}}{2} = \dfrac{2 \pm 2\sqrt{7}}{2} = 1 \pm \sqrt{7}$. Do not divide both sides by x, because that loses the solution $x = 0$.

Your turn · More techniques

(a) $x^3 + 3x^2 - 4x = 0$

$x = 0$, -4 or 1
$x(x^2 + 3x - 4) = x(x + 4)(x - 1) = 0$.

(b) $x^3 - 2x^2 - x + 2 = 0$ (group the terms in pairs)

$x = 2$, 1 or -1
$x^2(x - 2) - (x - 2) = (x - 2)(x^2 - 1) = (x - 2)(x - 1)(x + 1) = 0$. Take care with the sign when taking out -1 from $-x + 2$.

(c) $x^3 + 2x^2 - 2x = 0$

$x = 0$, $-1 + \sqrt{3}$ or $-1 - \sqrt{3}$
$x(x^2 + 2x - 2) = 0$. The quadratic does not factorise, so use the formula: $x = \dfrac{-2 \pm \sqrt{4 + 8}}{2} = \dfrac{-2 \pm 2\sqrt{3}}{2} = -1 \pm \sqrt{3}$.

Challenge · A quartic

Solve $x^4 + x^3 - 7x^2 - x + 6 = 0$.

$x = 1, -1, -3$ or 2

Let $P(x) = x^4 + x^3 - 7x^2 - x + 6$. $P(1) = 1 + 1 - 7 - 1 + 6 = 0$, so $(x - 1)$ is a factor.

$$
\begin{array}{rr|rrrr}
 & & x^3 & +2x^2 & -5x & -6 \\
\hline
x-1 & x^4 & +x^3 & -7x^2 & -x & +6 \\
 & -\,x^4 & -x^3 & & & \\
\hline
 & & 2x^3 & -7x^2 & & \\
 & -\,2x^3 & -2x^2 & & & \\
\hline
 & & & -5x^2 & -x & \\
 & -\,-5x^2 & +5x & & & \\
\hline
 & & & & -6x & +6 \\
 & -\,-6x & +6 & & & \\
\hline
 & & & & & 0
\end{array}
$$

The quotient is $Q(x) = x^3 + 2x^2 - 5x - 6$. $Q(-1) = -1 + 2 + 5 - 6 = 0$, so $(x + 1)$ is a factor of $Q(x)$.

$$
\begin{array}{rr|rrr}
 & & x^2 & +x & -6 \\
\hline
x+1 & x^3 & +2x^2 & -5x & -6 \\
 & -\,x^3 & +x^2 & & \\
\hline
 & & x^2 & -5x & \\
 & -\,x^2 & +x & & \\
\hline
 & & & -6x & -6 \\
 & -\,-6x & -6 & & \\
\hline
 & & & & 0
\end{array}
$$

The quotient is $x^2 + x - 6 = (x + 3)(x - 2)$. So $P(x) = (x - 1)(x + 1)(x + 3)(x - 2)$. A quartic can need two uses of the factor theorem and long division. Degree 4 and four zeros is the maximum.

Check in · Factorising

Let $P(x) = x^3 + x^2 - 10x + 8$.

(a) Show that $(x - 2)$ is a factor of $P(x)$.

(b) Hence solve $P(x) = 0$.

(c) What is the greatest number of zeros a polynomial of degree 4 can have?

(a) $P(2) = 0$.

(b) $x = 2, -4$ **or** 1.

(c) 4.

(a) $P(2) = 8 + 4 - 20 + 8 = 0$, so $(x - 2)$ is a factor.

$$
\begin{array}{rl|rrr}
 & & x^2 & +3x & -4 \\
\hline
x-2 & | & x^3 & +x^2 & -10x & +8 \\
 & - & x^3 & -2x^2 & & \\
\hline
 & & & 3x^2 & -10x & \\
 & - & & 3x^2 & -6x & \\
\hline
 & & & & -4x & +8 \\
 & - & & & -4x & +8 \\
\hline
 & & & & & 0
\end{array}
$$

(b) The quotient is $x^2 + 3x - 4 = (x + 4)(x - 1)$, so $P(x) = (x - 2)(x + 4)(x - 1)$.

(c) A polynomial of degree n has at most n zeros.

Exit ticket

1. Solve $(x - 5)(2x + 3)(x + 1) = 0$.

$x = 5, -\frac{3}{2}$ **or** -1

$2x + 3 = 0$ gives $x = -\frac{3}{2}$. Take care with the sign.

2. Solve $x^3 + x^2 - 4x - 4 = 0$.

$x = -1, 2$ **or** -2

$P(-1) = -1 + 1 + 4 - 4 = 0$, so $(x + 1)$ is a factor.

$$
\begin{array}{rl|rrr}
 & & x^2 & +0x & -4 \\
\hline
x+1 & | & x^3 & +x^2 & -4x & -4 \\
 & - & x^3 & +x^2 & & \\
\hline
 & & & 0x^2 & -4x & \\
 & - & & 0x^2 & +0x & \\
\hline
 & & & & -4x & -4 \\
 & - & & & -4x & -4 \\
\hline
 & & & & & 0
\end{array}
$$

The quotient is $x^2 - 4 = (x - 2)(x + 2)$. Grouping also works: $x^2(x + 1) - 4(x + 1) = (x + 1)(x^2 - 4)$.

Practice worksheet

Zeros from factors

1. Solve each equation.

(a) $(x+3)(x-7)(x-1)=0$ (b) $(2x-1)(3x+4)(x-6)=0$ (c) $x(2x+3)(x-4)=0$

> **(a)** $x=-3$, 7 **or** 1 **(b)** $x=\frac{1}{2}$, $-\frac{4}{3}$ **or** 6 **(c)** $x=0$, $-\frac{3}{2}$ **or** 4
> Equate each factor to zero. In (c) the factor x gives $x=0$.

2. Find the zeros of $P(x)=(x-4)(x^2+x-12)$.

> $x=4$, 3 **or** -4
> Factorise the quadratic first: $x^2+x-12=(x+4)(x-3)$. So $P(x)=(x-4)(x+4)(x-3)$.

Using the factor theorem

3. Solve $x^3-x^2-14x+24=0$.

> $x=2$, 3 **or** -4
> $P(2)=8-4-28+24=0$, so $(x-2)$ is a factor.
>
> $$\begin{array}{rl|rrr} & & x^2 & +x & -12 \\ \hline x-2 & | & x^3 & -x^2 & -14x & +24 \\ & - & x^3 & -2x^2 & & \\ \hline & & & x^2 & -14x & \\ & - & & x^2 & -2x & \\ \hline & & & & -12x & +24 \\ & - & & & -12x & +24 \\ \hline & & & & & 0 \end{array}$$
>
> The quotient is $x^2+x-12=(x+4)(x-3)$.

4. Solve $x^3+5x^2+2x-8=0$.

> $x=1$, -2 **or** -4
> $P(1)=1+5+2-8=0$, so $(x-1)$ is a factor.
>
> $$\begin{array}{rl|rrr} & & x^2 & +6x & +8 \\ \hline x-1 & | & x^3 & +5x^2 & +2x & -8 \\ & - & x^3 & -x^2 & & \\ \hline & & & 6x^2 & +2x & \\ & - & & 6x^2 & -6x & \\ \hline & & & & 8x & -8 \\ & - & & & 8x & -8 \\ \hline & & & & & 0 \end{array}$$
>
> The quotient is $x^2+6x+8=(x+2)(x+4)$.

5. Show that $x^3 + x^2 + x - 3 = 0$ has exactly one real solution.

> **The only real solution is $x = 1$.**
> $P(1) = 1 + 1 + 1 - 3 = 0$, so $(x - 1)$ is a factor.
>
> $$\begin{array}{rr|rrr} & & x^2 & +2x & +3 \\ \hline x-1 & x^3 & +x^2 & +x & -3 \\ - & x^3 & -x^2 & & \\ \hline & & 2x^2 & +x & \\ - & & 2x^2 & -2x & \\ \hline & & & 3x & -3 \\ - & & & 3x & -3 \\ \hline & & & & 0 \end{array}$$
>
> $P(x) = (x - 1)(x^2 + 2x + 3)$. The discriminant of $x^2 + 2x + 3$ is $4 - 12 = -8$, which is negative, so there are no other real solutions.

Other techniques

6. Solve each equation. Give exact answers.
(a) $x^3 - 6x^2 + 4x = 0$ (b) $2x^3 + 4x^2 - 6x = 0$

> **(a) $x = 0$ or $x = 3 \pm \sqrt{5}$ (b) $x = 0$, -3 or 1**
> (a) $x(x^2 - 6x + 4) = 0$. By the formula, $x = \dfrac{6 \pm \sqrt{20}}{2} = 3 \pm \sqrt{5}$. (b) $2x(x^2 + 2x - 3) = 2x(x + 3)(x - 1) = 0$.

7. Solve $x^3 + 4x^2 - 9x - 36 = 0$ by grouping the terms in pairs.

> **$x = -4$, 3 or -3**
> $x^2(x + 4) - 9(x + 4) = (x + 4)(x^2 - 9) = (x + 4)(x - 3)(x + 3)$.

8. Let $P(x) = x^4 - 2x^3 - 7x^2 + 8x + 12$.

(a) Show that $(x + 1)$ is a factor of $P(x)$.

(b) Divide $P(x)$ by $(x + 1)$.

(c) Hence solve $P(x) = 0$.

(d) What is the greatest number of zeros $P(x)$ could have?

(a) $P(-1) = 0$. **(b)** $P(x) = (x + 1)(x^3 - 3x^2 - 4x + 12)$. **(c)** $x = -1, 3, 2$ **or** -2. **(d)** 4.

(a) $P(-1) = 1 + 2 - 7 - 8 + 12 = 0$.

$$
\begin{array}{rrrrrr}
 & & x^3 & -3x^2 & -4x & +12 \\
\cline{3-6}
x+1 & \multicolumn{1}{|r}{x^4} & -2x^3 & -7x^2 & +8x & +12 \\
- & x^4 & +x^3 & & & \\
\cline{2-3}
 & & -3x^3 & -7x^2 & & \\
- & & -3x^3 & -3x^2 & & \\
\cline{3-4}
 & & & -4x^2 & +8x & \\
- & & & -4x^2 & -4x & \\
\cline{4-5}
 & & & & 12x & +12 \\
- & & & & 12x & +12 \\
\cline{5-6}
 & & & & & 0
\end{array}
$$

(c) Group the cubic: $x^2(x - 3) - 4(x - 3) = (x - 3)(x^2 - 4) = (x - 3)(x - 2)(x + 2)$. So $P(x) = (x + 1)(x - 3)(x - 2)(x + 2)$.