MCINTYRE MATHS · YEAR 10 MATHEMATICS

Worked solutions

Lesson 4 · Remainder and factor theorems

Warm-up · Evaluating polynomials

(a) If $P(x) = 3x^3 - x^2 + 2x - 5$, find $P(2)$ and $P(-1)$.

> **$P(2) = 19$ and $P(-1) = -11$.**
> $P(2) = 3(8) - 4 + 4 - 5 = 19$ and $P(-1) = 3(-1) - 1 - 2 - 5 = -11$. Put brackets around negative inputs so that $(-1)^2 = 1$, not -1.

(b) Write $29 \div 5$ as: dividend = divisor × quotient + remainder.

> $29 = 5 \times 5 + 4$
> This has the same shape as $P(x) = A(x)Q(x) + R(x)$ from the lesson on division.

(c) If $P(x) = x^3 + 4x^2 - x - 6$, find $P(-1)$.

> $P(-1) = -2$
> $P(-1) = -1 + 4 + 1 - 6 = -2$. This is the skill we use all lesson.

Starter · A pattern

(a) Use long division to divide $P(x)$ by $(x - 2)$. What is the remainder?

> **Quotient $x^2 + 5$, remainder 6.**

$$
\begin{array}{rr|rrr}
 & & x^2 & +0x & +5 \\
\hline
x-2 & x^3 & -2x^2 & +5x & -4 \\
 & -\,x^3 & -2x^2 & & \\
\hline
 & & 0x^2 & +5x & \\
 & -\, & 0x^2 & +0x & \\
\hline
 & & & 5x & -4 \\
 & - & & 5x & -10 \\
\hline
 & & & & 6
\end{array}
$$

> $x^3 - 2x^2 + 5x - 4 = (x - 2)(x^2 + 5) + 6$. The x term of the quotient is 0.

(b) Find $P(2)$. What do you notice?

> $P(2) = 8 - 8 + 10 - 4 = 6$, **the same as the remainder.**
> Substituting the value that makes the divisor zero gave the remainder, with no long division.

(c) Find $P(-1)$ and $P(1)$.

> **$P(-1) = -12$ and $P(1) = 0$.**
> $P(-1) = -1 - 2 - 5 - 4 = -12$ and $P(1) = 1 - 2 + 5 - 4 = 0$.

(d) Predict the remainders on division by $(x + 1)$ and $(x - 1)$.

> **Dividing by $(x + 1)$ gives remainder -12. Dividing by $(x - 1)$ gives remainder 0, so $(x - 1)$ is a factor.**
> Check by long division: $P(x) = (x + 1)(x^2 - 3x + 8) - 12$ and $P(x) = (x - 1)(x^2 - x + 4)$.

Example 1

(a) $x - 1$

> **The remainder is -3.**
> By the remainder theorem the remainder is $P(1) = 1 - 3 + 4 - 5 = -3$. A negative remainder is fine here.

(b) $x + 2$

> **The remainder is -33.**
> $x + 2 = 0$ gives $x = -2$, so the remainder is $P(-2) = -8 - 12 - 8 - 5 = -33$.

Your turn

(a) $x - 1$

> **The remainder is 3.**
> $P(1) = 1 - 5 + 3 + 4 = 3$.

(b) $x + 2$

> **The remainder is -30.**
> $P(-2) = -8 - 5(4) + 3(-2) + 4 = -8 - 20 - 6 + 4 = -30$.

Example 2

(a) $x - 2$

> **$P(2) = -9 \neq 0$, so $(x - 2)$ is not a factor.**
> $P(2) = 8 - 8 - 8 - 1 = -9$. The remainder is -9, not zero.

(b) $x + 1$

> $P(-1) = 0$, **so** $(x + 1)$ **is a factor.**
> $P(-1) = -1 - 2 + 4 - 1 = 0$. Finish with a sentence.

$$
\begin{array}{rr|rrr}
 & & x^2 & -3x & -1 \\
\hline
x+1 & x^3 & -2x^2 & -4x & -1 \\
- & x^3 & +x^2 & & \\
\hline
 & & -3x^2 & -4x & \\
- & & -3x^2 & -3x & \\
\hline
 & & & -x & -1 \\
- & & & -x & -1 \\
\hline
 & & & & 0
\end{array}
$$

> Long division gives $P(x) = (x + 1)(x^2 - 3x - 1)$.

Your turn

(a) $x + 1$

> $P(-1) = 18 \neq 0$, **so** $(x + 1)$ **is not a factor.**
> $P(-1) = -2 + 1 + 13 + 6 = 18$.

(b) $x - 2$

> $P(2) = 0$, **so** $(x - 2)$ **is a factor.**
> $P(2) = 16 + 4 - 26 + 6 = 0$. In fact $P(x) = (x - 2)(x + 3)(2x - 1)$, which we will find in the next lesson.

Example 3

Find the value of k if $P(x) = x^3 + 2x^2 + kx - 5$ has remainder -3 when divided by $(x + 2)$.

> $k = -1$
> The remainder is $P(-2) = -3$.
> $P(-2) = (-2)^3 + 2(-2)^2 + k(-2) - 5 = -8 + 8 - 2k - 5 = -5 - 2k$.
> So $-5 - 2k = -3$, giving $k = -1$.
> Check: $P(x) = x^3 + 2x^2 - x - 5$ and $P(-2) = -8 + 8 + 2 - 5 = -3$.

Your turn

Find the value of k such that $x^3 - 3x^2 + x + k$ has a remainder of 4 when divided by $(x - 3)$.

> $k = 1$
> $P(3) = 4$, so $27 - 27 + 3 + k = 4$, which gives $3 + k = 4$ and $k = 1$.

Check in · Division and the factor theorem

Let $P(x) = x^3 + 3x^2 - 13x + 6$.
(a) Show that $P(2) = 0$.
(b) Hence factorise $P(x)$ as $(x - 2)B(x)$, where $B(x)$ is a quadratic.

(a) $P(2) = 8 + 12 - 26 + 6 = 0$.
(b) $P(x) = (x - 2)(x^2 + 5x - 3)$.

$$
\begin{array}{rr|rrr}
 & & x^2 & +5x & -3 \\
\hline
x-2 & x^3 & +3x^2 & -13x & +6 \\
- & x^3 & -2x^2 & & \\
\hline
 & & 5x^2 & -13x & \\
 & - & 5x^2 & -10x & \\
\hline
 & & & -3x & +6 \\
 & - & & -3x & +6 \\
\hline
 & & & & 0
\end{array}
$$

By the factor theorem, $(x - 2)$ is a factor. Long division of $P(x)$ by $(x - 2)$ gives the quotient $x^2 + 5x - 3$.
Check: $(x - 2)(x^2 + 5x - 3) = x^3 + 5x^2 - 3x - 2x^2 - 10x + 6 = x^3 + 3x^2 - 13x + 6$.

Challenge · Two unknowns

The polynomial $P(x) = x^3 + cx^2 + dx - 6$ is divisible by both $(x - 2)$ and $(x + 1)$. Find the values of c and d.

$c = 2$ and $d = -5$
$P(2) = 0$ gives $8 + 4c + 2d - 6 = 0$, so $2c + d = -1$.
$P(-1) = 0$ gives $-1 + c - d - 6 = 0$, so $c - d = 7$.
Adding: $3c = 6$, so $c = 2$ and $d = -5$.
Check: $P(x) = x^3 + 2x^2 - 5x - 6 = (x - 2)(x + 1)(x + 3)$.

Exit ticket

1. Find the remainder when $x^3 + x^2 - 4x + 3$ is divided by $(x + 3)$.

The remainder is -3.
$P(-3) = -27 + 9 + 12 + 3 = -3$. A common slip is to use $P(3)$.

2. Is $(x - 3)$ a factor of $x^3 - 2x^2 - 5x + 6$? Show your reasoning.

Yes. $P(3) = 0$, so $(x - 3)$ is a factor.
$P(3) = 27 - 18 - 15 + 6 = 0$. Always finish with a concluding sentence.

Practice worksheet

Remainders

1. Find the remainder when $P(x) = x^3 + 2x^2 - 5x + 4$ is divided by: (a) $x - 1$ (b) $x + 3$ (c) $x - 2$.

> **(a)** 2 **(b)** 10 **(c)** 10
> $P(1) = 1 + 2 - 5 + 4 = 2$, $P(-3) = -27 + 18 + 15 + 4 = 10$, $P(2) = 8 + 8 - 10 + 4 = 10$.

2. Find the remainder when $P(x) = x^4 - 3x^2 + x - 2$ is divided by: (a) $x - 2$ (b) $x + 1$.

> **(a)** 4 **(b)** -5
> $P(2) = 16 - 12 + 2 - 2 = 4$ and $P(-1) = 1 - 3 - 1 - 2 = -5$.

3. What is the remainder when $2x^3 - x^2 + 4$ is divided by $(x + 2)$?

> -16
> $P(-2) = -16 - 4 + 4 = -16$.

Factors

4. Which of these are factors of $P(x) = x^3 - x^2 - 10x - 8$? (a) $x + 1$ (b) $x - 2$ (c) $x - 4$ (d) $x + 2$

> **(a), (c) and (d) are factors; (b) is not.**
> $P(-1) = 0$, $P(2) = -24$, $P(4) = 0$, $P(-2) = 0$. In fact $P(x) = (x + 1)(x - 4)(x + 2)$.

5. Show that $(x - 3)$ is a factor of $P(x) = x^3 - 4x^2 - 3x + 18$. Hence write $P(x) = (x - 3)Q(x)$, where $Q(x)$ is a quadratic.

> $P(3) = 0$ **and** $P(x) = (x - 3)(x^2 - x - 6)$.
> $P(3) = 27 - 36 - 9 + 18 = 0$.
>
> $$\begin{array}{rr|rrr} & & x^2 & -x & -6 \\ \hline x-3 & x^3 & -4x^2 & -3x & +18 \\ - & x^3 & -3x^2 & & \\ \hline & & -x^2 & -3x & \\ - & & -x^2 & +3x & \\ \hline & & & -6x & +18 \\ - & & & -6x & +18 \\ \hline & & & & 0 \end{array}$$
>
> Long division gives $Q(x) = x^2 - x - 6 = (x - 3)(x + 2)$, so $P(x) = (x - 3)^2(x + 2)$.

Unknown coefficients

6. Find k if $x^3 - kx^2 + 2x + 5$ has remainder 3 when divided by $(x - 1)$.

> $k = 5$
> $P(1) = 1 - k + 2 + 5 = 8 - k = 3$, so $k = 5$.

7. Find k if $(x + 2)$ is a factor of $x^3 + 4x^2 + kx + 8$.

> $k = 8$
> $P(-2) = -8 + 16 - 2k + 8 = 16 - 2k = 0$, so $k = 8$.

8. The polynomial $P(x) = x^3 + ax^2 + bx - 12$ is divisible by $(x - 2)$ and leaves remainder 6 when divided by $(x + 1)$. Find a and b.

> **$a = 7$ and $b = -12$**
> $P(2) = 0$ gives $8 + 4a + 2b - 12 = 0$, so $2a + b = 2$.
> $P(-1) = 6$ gives $-1 + a - b - 12 = 6$, so $a - b = 19$.
> Adding: $3a = 21$, so $a = 7$ and $b = -12$.
> Check: $P(x) = x^3 + 7x^2 - 12x - 12$, $P(2) = 8 + 28 - 24 - 12 = 0$ and $P(-1) = -1 + 7 + 12 - 12 = 6$.