MCINTYRE MATHS · YEAR 10 MATHEMATICS

Worked solutions

Lesson 3 · Multiplying and dividing polynomials

Warm-up · Expanding

(a) Expand $3x(2x - 5)$.

> $6x^2 - 15x$
> Multiply each term inside the bracket by $3x$.

(b) Expand and simplify $(x + 4)(x - 3)$.

> $x^2 + x - 12$
> $x^2 - 3x + 4x - 12 = x^2 + x - 12$.

(c) Expand $(2x - 1)^2$.

> $4x^2 - 4x + 1$
> $(2x - 1)(2x - 1) = 4x^2 - 2x - 2x + 1$. Squaring a bracket means multiplying it by itself. It is not $4x^2 + 1$.

Example 1 · Expand and simplify

(a) $x^3(2x^2 - 3x)$

> $2x^5 - 3x^4$
> $x^3 \times 2x^2 = 2x^5$ and $x^3 \times (-3x) = -3x^4$.

(b) $(x + 3)(x^3 - 2x + 5)$

> $x^4 + 3x^3 - 2x^2 - x + 15$
> $x^4 - 2x^2 + 5x + 3x^3 - 6x + 15$. There are $2 \times 3 = 6$ terms before simplifying, and the degree is $1 + 3 = 4$.

Example 2 · Products of polynomials

(a) $P(x) \times Q(x)$

> $x^5 - x^4 + 5x^3 - 3x^2 + 7x - 3$
> $x^5 - x^4 + 3x^3 + 2x^3 - 2x^2 + 6x - x^2 + x - 3$. There are 9 terms before simplifying.

(b) $(P(x))^2$

> $x^6 + 4x^4 - 2x^3 + 4x^2 - 4x + 1$
> $(x^3 + 2x - 1)(x^3 + 2x - 1)$. Multiply the polynomial by itself. Do not square each term separately.

Your turn · Expand and simplify

(a) $2x(x^2 - 3x + 4)$

> $2x^3 - 6x^2 + 8x$
> Multiply each term by $2x$.

(b) $(x + 3)(x^2 - 2x + 5)$

> $x^3 + x^2 - x + 15$
> $x^3 - 2x^2 + 5x + 3x^2 - 6x + 15$.

(c) $(2x - 1)(x^2 + x - 3)$

> $2x^3 + x^2 - 7x + 3$
> $2x^3 + 2x^2 - 6x - x^2 - x + 3$.

(d) $(x^2 + 2)^2$

> $x^4 + 4x^2 + 4$
> $(x^2 + 2)(x^2 + 2) = x^4 + 2x^2 + 2x^2 + 4$. It is not $x^4 + 4$.

Example 3 · Long division

Divide $P(x) = x^3 + 2x^2 - 5x + 1$ by $(x - 2)$ and write in the form $P(x) = (x - 2)Q(x) + R(x)$, where $R(x)$ is the remainder.

> **Quotient $x^2 + 4x + 3$, remainder 7.**
> $x^3 + 2x^2 - 5x + 1 = (x - 2)(x^2 + 4x + 3) + 7$

		x^2	$+4x$	$+3$
$x - 2$	x^3	$+2x^2$	$-5x$	$+1$
$-$	x^3	$-2x^2$		
		$4x^2$	$-5x$	
$-$		$4x^2$	$-8x$	
			$3x$	$+1$
$-$			$3x$	-6
				7

> $x^3 \div x = x^2$. Subtract $x^3 - 2x^2$ to leave $4x^2 - 5x$. Then $4x^2 \div x = 4x$. Subtract $4x^2 - 8x$ to leave $3x + 1$. Then $3x \div x = 3$. Subtract $3x - 6$ to leave 7, which has lower degree than $x - 2$, so we stop. Check: $(x - 2)(x^2 + 4x + 3) + 7$ expands back to $P(x)$.

Example 4 · A negative remainder

Divide $P(x) = 4x^3 - x^2 + 6x - 5$ by $(x + 1)$ and write in the form $P(x) = (x + 1)Q(x) + R(x)$.

Quotient $4x^2 - 5x + 11$, remainder -16.
$4x^3 - x^2 + 6x - 5 = (x + 1)(4x^2 - 5x + 11) - 16$

		$4x^2$	$-5x$	$+11$
$x+1$	$4x^3$	$-x^2$	$+6x$	-5
-	$4x^3$	$+4x^2$		
		$-5x^2$	$+6x$	
-		$-5x^2$	$-5x$	
			$11x$	-5
-			$11x$	$+11$
				-16

$4x^3 \div x = 4x^2$. Subtract $4x^3 + 4x^2$ to leave $-5x^2 + 6x$. Then $-5x^2 \div x = -5x$. Subtract $-5x^2 - 5x$ to leave $11x - 5$. Then $11x \div x = 11$. Subtract $11x + 11$ to leave -16. A remainder can be negative.

Your turn · Long division

(a) $x^3 - 2x^2 + x - 5$ by $(x - 3)$

$x^3 - 2x^2 + x - 5 = (x - 3)(x^2 + x + 4) + 7$

		x^2	$+x$	$+4$
$x-3$	x^3	$-2x^2$	$+x$	-5
-	x^3	$-3x^2$		
		x^2	$+x$	
-		x^2	$-3x$	
			$4x$	-5
-			$4x$	-12
				7

Quotient $x^2 + x + 4$, remainder 7.

(b) $2x^3 + x^2 - 7x + 5$ by $(x + 2)$

$2x^3 + x^2 - 7x + 5 = (x + 2)(2x^2 - 3x - 1) + 7$

$$
\begin{array}{rr|llll}
 & & 2x^2 & -3x & -1 & \\
\hline
x+2 & 2x^3 & +x^2 & -7x & +5 & \\
 & -\ 2x^3 & +4x^2 & & & \\
\hline
 & & -3x^2 & -7x & & \\
 & - & -3x^2 & -6x & & \\
\hline
 & & & -x & +5 & \\
 & - & & -x & -2 & \\
\hline
 & & & & 7 &
\end{array}
$$

Quotient $2x^2 - 3x - 1$, remainder 7.

(c) $x^3 - 7x + 6$ by $(x - 2)$

$x^3 - 7x + 6 = (x - 2)(x^2 + 2x - 3)$

$$
\begin{array}{rr|llll}
 & & x^2 & +2x & -3 & \\
\hline
x-2 & x^3 & +0x^2 & -7x & +6 & \\
 & -\ x^3 & -2x^2 & & & \\
\hline
 & & 2x^2 & -7x & & \\
 & - & 2x^2 & -4x & & \\
\hline
 & & & -3x & +6 & \\
 & - & & -3x & +6 & \\
\hline
 & & & & 0 &
\end{array}
$$

Write $x^3 + 0x^2 - 7x + 6$. The quotient is $x^2 + 2x - 3$ and the remainder is 0, so $(x - 2)$ divides $P(x)$ exactly. A remainder of 0 means $(x - 2)$ is a factor.

Check in · Division and factors

Let $P(x) = x^3 + 2x^2 - 5x - 6$.
(i) Show that $P(2) = 0$.
(ii) Hence write $P(x)$ as $(x-2)B(x)$, where $B(x)$ is a quadratic polynomial.

(i) $P(2) = 0$.
(ii) $P(x) = (x-2)(x^2+4x+3)$.

$$
\begin{array}{rrrrr}
 & & x^2 & +4x & +3 \\
\cline{3-5}
x-2 & \multicolumn{1}{|r}{x^3} & +2x^2 & -5x & -6 \\
 - & x^3 & -2x^2 & & \\
\cline{2-3}
 & & 4x^2 & -5x & \\
 - & & 4x^2 & -8x & \\
\cline{3-4}
 & & & 3x & -6 \\
 - & & & 3x & -6 \\
\cline{4-5}
 & & & & 0
\end{array}
$$

(i) $P(2) = 8 + 8 - 10 - 6 = 0$.
(ii) Divide by $(x-2)$. The quotient is x^2+4x+3 and the remainder is 0. It factorises further as $(x-2)(x+1)(x+3)$.

Exit ticket

1. Expand and simplify $(x-3)(x^2+2x-1)$.

$x^3 - x^2 - 7x + 3$
$x^3 + 2x^2 - x - 3x^2 - 6x + 3 = x^3 - x^2 - 7x + 3$.

2. Divide $2x^3 - 3x^2 + x + 4$ by $(x-1)$ and write in the form $P(x) = A(x)Q(x) + R(x)$.

$2x^3 - 3x^2 + x + 4 = (x-1)(2x^2 - x) + 4$

$$
\begin{array}{rrrrr}
 & & 2x^2 & -x & +0 \\
\cline{3-5}
x-1 & \multicolumn{1}{|r}{2x^3} & -3x^2 & +x & +4 \\
 - & 2x^3 & -2x^2 & & \\
\cline{2-3}
 & & -x^2 & +x & \\
 - & & -x^2 & +x & \\
\cline{3-4}
 & & & 0x & +4 \\
 - & & & 0x & +0 \\
\cline{4-5}
 & & & & 4
\end{array}
$$

Quotient $2x^2 - x$, remainder 4. The quotient has no constant term.

Practice worksheet

Expanding

1. Expand and simplify.
(a) $3x(x^2 - 2x + 5)$ (b) $(x - 4)(x^2 + 3x - 2)$ (c) $(2x + 1)(x^3 - x + 3)$

> **(a)** $3x^3 - 6x^2 + 15x$ **(b)** $x^3 - x^2 - 14x + 8$ **(c)** $2x^4 + x^3 - 2x^2 + 5x + 3$
> (b) $x^3 + 3x^2 - 2x - 4x^2 - 12x + 8$. (c) $2x^4 - 2x^2 + 6x + x^3 - x + 3$.

2. Let $P(x) = x^2 - 2x + 3$ and $Q(x) = x^3 + x - 1$. Expand and simplify $P(x) \times Q(x)$.

> $x^5 - 2x^4 + 4x^3 - 3x^2 + 5x - 3$
> There are $3 \times 3 = 9$ terms before simplifying: $x^5 + x^3 - x^2 - 2x^4 - 2x^2 + 2x + 3x^3 + 3x - 3$.

3. Expand and simplify $(x^2 - x + 1)^2$.

> $x^4 - 2x^3 + 3x^2 - 2x + 1$
> Multiply $(x^2 - x + 1)$ by itself. It is not $x^4 + x^2 + 1$.

4. Consider $(x^2 + x + 1)(x^3 + x^2 + x + 1)$.
(a) How many terms are there before simplifying?
(b) What is the degree of the product?
(c) What is the greatest number of terms the simplified answer could have?

> **(a)** 12 **(b)** 5 **(c)** 6
> (a) $3 \times 4 = 12$. (b) $2 + 3 = 5$. (c) A polynomial of degree 5 has at most $5 + 1 = 6$ terms. The product simplifies to $x^5 + 2x^4 + 3x^3 + 3x^2 + 2x + 1$, which has exactly 6.

Dividing

5. Divide and write in the form $P(x) = A(x)Q(x) + R(x)$.
(a) $x^3 + 4x^2 - x - 6$ by $(x - 1)$
(b) $2x^3 - 5x^2 + 3x - 7$ by $(x - 2)$

> **(a)** $(x - 1)(x^2 + 5x + 4) - 2$ **(b)** $(x - 2)(2x^2 - x + 1) - 5$
>
> $$\begin{array}{rl|llll} & & x^2 & +5x & +4 \\ \hline x-1 & x^3 & +4x^2 & -x & -6 \\ & - & x^3 & -x^2 & \\ \hline & & 5x^2 & -x & \\ & - & 5x^2 & -5x & \\ \hline & & & 4x & -6 \\ & - & & 4x & -4 \\ \hline & & & & -2 \end{array}$$
>
> (a) Quotient $x^2 + 5x + 4$, remainder -2. (b) Quotient $2x^2 - x + 1$, remainder -5.

6. Divide $x^3 - 8$ by $(x - 2)$. Write the missing terms with zero coefficients.

$x^3 - 8 = (x - 2)(x^2 + 2x + 4)$

$$
\begin{array}{rrrrr}
 & & x^2 & +2x & +4 \\
\cline{2-5}
x-2 & x^3 & +0x^2 & +0x & -8 \\
- & x^3 & -2x^2 & & \\
\cline{2-3}
 & & 2x^2 & +0x & \\
- & & 2x^2 & -4x & \\
\cline{3-4}
 & & & 4x & -8 \\
- & & & 4x & -8 \\
\cline{4-5}
 & & & & 0
\end{array}
$$

Write $x^3 + 0x^2 + 0x - 8$. The quotient is $x^2 + 2x + 4$ and the remainder is 0.

7. Divide $3x^3 + 2x^2 - 5x + 4$ by $(x + 1)$ and write in the form $P(x) = (x + 1)Q(x) + R(x)$.

$3x^3 + 2x^2 - 5x + 4 = (x + 1)(3x^2 - x - 4) + 8$

$$
\begin{array}{rrrrr}
 & & 3x^2 & -x & -4 \\
\cline{2-5}
x+1 & 3x^3 & +2x^2 & -5x & +4 \\
- & 3x^3 & +3x^2 & & \\
\cline{2-3}
 & & -x^2 & -5x & \\
- & & -x^2 & -x & \\
\cline{3-4}
 & & & -4x & +4 \\
- & & & -4x & -4 \\
\cline{4-5}
 & & & & 8
\end{array}
$$

Quotient $3x^2 - x - 4$, remainder 8.

8. Let $P(x) = x^3 - 6x^2 + 11x - 6$.
(i) Show that $P(1) = 0$.
(ii) Divide $P(x)$ by $(x - 1)$.
(iii) Hence write $P(x)$ as a product of three linear factors.

(i) $P(1) = 0$. **(ii)** $P(x) = (x - 1)(x^2 - 5x + 6)$. **(iii)** $P(x) = (x - 1)(x - 2)(x - 3)$.

$$
\begin{array}{rrrrr}
 & & x^2 & -5x & +6 \\
\cline{2-5}
x-1 & x^3 & -6x^2 & +11x & -6 \\
- & x^3 & -x^2 & & \\
\cline{2-3}
 & & -5x^2 & +11x & \\
- & & -5x^2 & +5x & \\
\cline{3-4}
 & & & 6x & -6 \\
- & & & 6x & -6 \\
\cline{4-5}
 & & & & 0
\end{array}
$$

(i) $1 - 6 + 11 - 6 = 0$. (ii) The quotient is $x^2 - 5x + 6$ and the remainder is 0. (iii) $x^2 - 5x + 6 = (x - 2)(x - 3)$.