MCINTYRE MATHS · YEAR 10 MATHEMATICS

Worked solutions

Lesson 2 · Introducing polynomials

Warm-up · Indices and substitution

(a) Simplify $x^2 \times x^3$.

> x^5
> Add the indices when multiplying powers of the same base.

(b) Write $\dfrac{1}{x}$ and $\sqrt{x}$ as powers of x.

> $\dfrac{1}{x} = x^{-1}$ **and** $\sqrt{x} = x^{1/2}$
> This is the key idea for today. A polynomial only has whole-number powers, so a negative or fractional power rules an expression out.

(c) Evaluate $2x^3 - x$ when $x = -2$.

> -14
> $2(-8) - (-2) = -16 + 2 = -14$. Put brackets around the negative input.

Decide and state the degree

(a) $6x - x^4 + 2x^2$

> **Yes. Degree 4 (quartic).**
> In order it is $-x^4 + 2x^2 + 6x$. All the powers are whole numbers. The degree is the highest power, not the power of the first term written.

(b) $x^2 + \sqrt{x} - 3$

> **No.**
> $\sqrt{x} = x^{1/2}$, which is not a whole-number power.

Identify and describe

(a) $y = x^5 - 6x^4 + 3x^2 - x + 6$

> **Polynomial. Degree 5, leading coefficient 1 (monic), constant term 6.**
> The highest power is 5 and its coefficient is 1. The term with no x is 6.

(b) $P(x) = 7x^7 - 18x^3 + x^4 - 6$

> **Polynomial. Degree 7, leading coefficient 7, constant term -6.**
> The terms are out of order. Rewritten, $P(x) = 7x^7 + x^4 - 18x^3 - 6$. Missing powers such as x^6 and x are allowed.

(c) $f(x) = x^4 - x^{-2} + \sqrt{x} - \dfrac{1}{2}$

> **Not a polynomial.**
> It has x^{-2} and $\sqrt{x} = x^{1/2}$.

(d) $g(x) = \dfrac{1}{x^2} - 6x + \dfrac{1}{\sqrt{x}}$

> **Not a polynomial.**
> $\dfrac{1}{x^2} = x^{-2}$ and $\dfrac{1}{\sqrt{x}} = x^{-1/2}$.

Your turn

(a) $7x^4 - x^2 + 9$

> **Polynomial. Degree 4, leading coefficient 7.**
> The highest power is 4 and its coefficient is 7.

(b) $2x + \dfrac{3}{x}$

> **Not a polynomial.**
> $\dfrac{3}{x} = 3x^{-1}$.

(c) $(x^2 + 1)(x - 3)$

> **Polynomial. Degree 3, leading coefficient 1.**
> Expanded it is $x^3 - 3x^2 + x - 3$. A product of polynomials is a polynomial, and the degree is $2 + 1 = 3$.

(d) $x^2 + \sqrt{x}$

> **Not a polynomial.**
> $\sqrt{x} = x^{1/2}$.

Example 3 · Finding P(a)

(a) $P(2)$

> $P(2) = 21$
> $2^3 + 3(2^2) - 2(2) + 5 = 8 + 12 - 4 + 5 = 21$.

(b) $P(-1)$

> $P(-1) = 9$
> $(-1)^3 + 3(-1)^2 - 2(-1) + 5 = -1 + 3 + 2 + 5 = 9$.

Example 4 · Combining polynomials

(a) $P(1) + Q(2)$

> 2
> $P(1) = 1 - 1 + 2 = 2$ and $Q(2) = 4 - 4 = 0$.

(b) $P(-1) - Q(-1)$

> -7
> $P(-1) = -1 - 1 - 2 = -4$ and $Q(-1) = 4 - 1 = 3$, so $-4 - 3 = -7$.

(c) $P(x) + Q(x)$, simplified

> $x^3 - 2x^2 + 2x + 4$
> $x^3 - x^2 + 2x + 4 - x^2$. Collect the like terms.

(d) $P(x) - Q(x)$, simplified

> $x^3 + 2x - 4$
> $x^3 - x^2 + 2x - (4 - x^2) = x^3 - x^2 + 2x - 4 + x^2$. Put brackets around $Q(x)$ so that every sign changes.

Challenge · An unknown coefficient

The polynomial $P(x) = x^3 - 4x^2 + bx - 2$ has $P(1) = 5$. Find the value of b.

> $b = 10$
> $P(1) = 1 - 4 + b - 2 = b - 5$. Setting $b - 5 = 5$ gives $b = 10$. Using a known value of P to find an unknown coefficient is the idea behind the factor and remainder theorems.

Check in · Polynomials

Let $P(x) = 2x^3 - 5x^2 + x - 7$.
(a) State the degree, leading coefficient and constant term.
(b) Find $P(-2)$.
(c) Explain why $x^3 + \dfrac{1}{x}$ is not a polynomial.

> **(a) Degree 3, leading coefficient 2, constant term -7.**
> **(b) $P(-2) = -45$.**
> **(c) It contains x^{-1}.**
> (b) $2(-8) - 5(4) + (-2) - 7 = -16 - 20 - 2 - 7 = -45$.
> (c) $\dfrac{1}{x} = x^{-1}$, and a polynomial only has whole-number powers $0, 1, 2, \ldots$

Exit ticket

1. Is $3x^2 - \sqrt{x}$ a polynomial? Explain.

> **No.**
> $\sqrt{x} = x^{1/2}$, which is not a whole-number power.

2. For $P(x) = x^4 - 2x + 1$, state the degree, say whether $P(x)$ is monic, and find $P(-1)$.

> **Degree 4. Monic. $P(-1) = 4$.**
> The leading coefficient is 1, so it is monic. $P(-1) = 1 + 2 + 1 = 4$. Take care with the sign on $-2x$.

Practice worksheet

Identifying polynomials

1. Decide which are polynomials. State the degree of those that are.
(a) $3x^5 - x + 2$ (b) $\dfrac{5}{x} + x$ (c) $x^2 + \sqrt{3}\,x - 1$ (d) $2x^4 - \sqrt{x}$

> **(a) polynomial, degree 5 (b) not a polynomial (c) polynomial, degree 2 (d) not a polynomial**
> (b) $\dfrac{5}{x} = 5x^{-1}$. (c) $\sqrt{3}$ is just a number, so it is an ordinary coefficient. (d) $\sqrt{x} = x^{1/2}$.

2. For $P(x) = -4x^3 + x^5 - 2x^2 + 9$, write the terms in descending order, then state the degree, leading term, leading coefficient and constant term. Is $P(x)$ monic?

> **$P(x) = x^5 - 4x^3 - 2x^2 + 9$. Degree 5, leading term x^5, leading coefficient 1, constant term 9. It is monic.**
> The leading coefficient is 1, so $P(x)$ is monic.

3. Name each polynomial by its degree.
(a) 9 (b) $5 - 2x$ (c) $1 - x^2 + x^3$ (d) $x^4 + x - 2$

> **(a) constant (b) linear (c) cubic (d) quartic**
> The degrees are 0, 1, 3 and 4.

Evaluating polynomials

4. If $P(x) = x^3 - 3x^2 + x - 5$, find $P(0)$, $P(1)$, $P(3)$ and $P(-2)$.

> **$P(0) = -5$, $P(1) = -6$, $P(3) = -2$, $P(-2) = -27$**
> $P(3) = 27 - 27 + 3 - 5 = -2$ and $P(-2) = -8 - 12 - 2 - 5 = -27$.

5. If $Q(x) = 2x^3 + x^2 - 4$, find $Q(2)$, $Q(-1)$ and $Q\left(\frac{1}{2}\right)$.

> **$Q(2) = 16$, $Q(-1) = -5$, $Q\left(\frac{1}{2}\right) = -\frac{7}{2}$**
> $Q(2) = 16 + 4 - 4 = 16$. $Q(-1) = -2 + 1 - 4 = -5$. $Q\left(\frac{1}{2}\right) = \frac{1}{4} + \frac{1}{4} - 4 = -\frac{7}{2}$.

Combining and unknowns

6. Let $P(x) = x^2 + 3x - 1$ and $Q(x) = 2x^2 - x + 4$. Simplify (a) $P(x) + Q(x)$ (b) $P(x) - Q(x)$ (c) $2P(x) - Q(x)$.

> **(a) $3x^2 + 2x + 3$ (b) $-x^2 + 4x - 5$ (c) $7x - 6$**
> (c) $2x^2 + 6x - 2 - 2x^2 + x - 4 = 7x - 6$. The degree drops because the x^2 terms cancel.

7. The polynomial $P(x) = x^3 + kx^2 - 3x + 2$ has $P(2) = 12$. Find k.

> **$k = 2$**
> $P(2) = 8 + 4k - 6 + 2 = 4 + 4k$. Setting $4 + 4k = 12$ gives $k = 2$.

8. The polynomial $P(x) = x^3 + ax^2 + bx + 6$ has $P(1) = 0$ and $P(-1) = 10$.

(a) Find a and b.

(b) State the degree of $P(x)$ and say whether it is monic.

(c) Find $P(2)$.

> **(a) $a = -1$, $b = -6$. (b) Degree 3, monic. (c) $P(2) = -2$.**
> (a) $P(1) = 7 + a + b = 0$ gives $a + b = -7$. $P(-1) = 5 + a - b = 10$ gives $a - b = 5$. Adding, $2a = -2$, so $a = -1$ and $b = -6$. (c) $P(x) = x^3 - x^2 - 6x + 6$, so $P(2) = 8 - 4 - 12 + 6 = -2$.