MCINTYRE MATHS · YEAR 10 MATHEMATICS

Worked solutions

Lesson 1 · Relations and functions

Warm-up · Substitution

(a) Evaluate $3x - 4$ when $x = -2$.

> -10
> $3(-2) - 4 = -6 - 4 = -10$.

(b) Evaluate $x^2 - x + 6$ when $x = -3$.

> 18
> $(-3)^2 - (-3) + 6 = 9 + 3 + 6 = 18$. Put brackets around a negative input so that $(-3)^2 = 9$, not -9.

(c) Solve $x^2 = 9$.

> $x = 3$ **or** $x = -3$
> Both 3^2 and $(-3)^2$ equal 9. One equation, two answers: this idea returns when we meet one-to-many relations.

Classify each relation

(a) $y = x - 5$

> **One-to-one.**
> Each x gives one y, and each y comes from exactly one x.

(b) $y = x^2 + 3$

> **Many-to-one.**
> $x = 2$ and $x = -2$ both give $y = 7$, but each x gives only one y.

(c) $x = y^2 + 1$

> **One-to-many.**
> $x = 5$ gives $y = 2$ or $y = -2$, while each y gives only one x.

(d) $x^2 + y^2 = 25$

> **Many-to-many.**
> $x = 3$ gives $y = 4$ or $y = -4$, and $y = 4$ gives $x = 3$ or $x = -3$. Test with a number: choose an x and count the y values, then choose a y and count the x values.

Identify the functions

(a) $\{(-2, 3), (0, 3), (1, -4), (5, 0)\}$

> **Function (many-to-one).**
> Each x appears once. The repeated $y = 3$ is allowed. Only a repeated x with different y values would break the rule.

(b) $y = 4 - x^2$

> **Function (many-to-one).**
> Each x gives exactly one y.

(c) The graph shown after part (c).

> **Not a function.**
> The graph is $x = y^2 - 1$. The vertical line $x = 3$ crosses it twice, at $y = 2$ and $y = -2$, so one input has two outputs.

Your turn

(a) $\{(1, 3), (2, 5), (3, 3), (4, 7)\}$

> **Function.**
> Each x appears once. The repeated $y = 3$ is allowed.

(b) $\{(0, 1), (2, 4), (0, -3)\}$

> **Not a function.**
> $x = 0$ gives both $y = 1$ and $y = -3$.

(c) $y = x^2 - 2$

> **Function (many-to-one).**
> Each x gives one y.

(d) $x = 4$

> **Not a function.**
> The graph is a vertical line, so the single input $x = 4$ has infinitely many outputs. The equation has no y, so it is easy to miss.

Evaluate

(a) $f(0)$

> $f(0) = -4$
> $0 + 0 - 4 = -4.$

(b) $f(-2)$

> $f(-2) = -6$
> $(-2)^2 + 3(-2) - 4 = 4 - 6 - 4 = -6.$

(c) $f(k)$

> $f(k) = k^2 + 3k - 4$
> Replace every x with k.

(d) $f(x+1)$

> $f(x+1) = x^2 + 5x$
> $(x+1)^2 + 3(x+1) - 4 = x^2 + 2x + 1 + 3x + 3 - 4 = x^2 + 5x$. Keep brackets around $x+1$, and note that $f(x+1)$ is not $f(x) + 1$.

Your turn

(a) $g(0)$

> -5
> $g(0) = -5$.

(b) $g(-1)$

> -4
> $3(1) - 2 - 5 = -4$.

(c) $g(a)$

> $3a^2 + 2a - 5$
> Replace every x with a.

(d) $g(a+1)$

> $3a^2 + 8a$
> $3(a+1)^2 + 2(a+1) - 5 = 3a^2 + 6a + 3 + 2a + 2 - 5 = 3a^2 + 8a$. Expand $(a+1)^2$ fully as $a^2 + 2a + 1$.

Domain and range

(a) $y = 5 - 2x$

> **Domain: all real x. Range: all real y.**
> A straight line carries on forever in both directions.

(b) $y = -x^2 + 4$

> **Domain: all real x. Range: $y \leq 4$.**
> $-x^2$ is never positive, so the largest value of y is 4, at $x = 0$. Sketch the downward parabola first.

(c) $y = \dfrac{1}{x}$

> **Domain: $x \neq 0$. Range: $y \neq 0$.**
> We cannot divide by zero, and 1 divided by a number is never 0.

Check in · Functions

Let $f(x) = 3 - x^2$.
(a) Find $f(-2)$.
(b) Write down the domain and range of f.
(c) Is f one-to-one? Explain.

> **(a) $f(-2) = -1$.**
> **(b) Domain: all real x. Range: $y \leq 3$.**
> **(c) No.**
> (a) $3 - (-2)^2 = 3 - 4 = -1$.
> (b) $-x^2 \leq 0$, so $f(x) \leq 3$.
> (c) $f(-2) = -1$ and $f(2) = -1$, so two inputs give the same output. It is many-to-one, and the horizontal line $y = -1$ crosses the graph twice.

Exit ticket

1. Is $x^2 + y^2 = 9$ a function? Explain.

> **No.**
> $x = 0$ gives $y = 3$ and $y = -3$. A circle fails the vertical line test (it is many-to-many).

2. If $h(x) = 2x^2 - 3$, find $h(-3)$ and state the range.

> $h(-3) = 15$. **Range: $y \geq -3$.**
> $2(9) - 3 = 15$. The smallest value of $2x^2$ is 0, so $h(x) \geq -3$.

Practice worksheet

Relations and functions

1. Say whether each relation is one-to-one, one-to-many, many-to-one or many-to-many.
(a) $y = 3x - 2$ (b) $y = x^2 - 4$ (c) $x = y^2 + 2$ (d) $x^2 + y^2 = 16$

> **(a) one-to-one (b) many-to-one (c) one-to-many (d) many-to-many**
> Test with numbers. (b) $x = 2$ and $x = -2$ both give $y = 0$. (c) $x = 6$ gives $y = 2$ or $y = -2$. (d) $x = 0$ gives $y = 4$ or $y = -4$, and $y = 0$ gives $x = 4$ or $x = -4$.

2. Which of these are functions?
(a) $\{(0, 2), (1, 5), (2, 2), (3, 8)\}$ (b) $\{(1, 4), (1, 7), (2, 3)\}$ (c) $y = x^3$ (d) $x = -2$

> **(a) and (c) are functions. (b) and (d) are not.**
> In (b) $x = 1$ has two outputs. In (d) the graph is a vertical line.

Function notation

3. For $f(x) = 2x^2 - x + 3$, find $f(0)$, $f(-1)$ and $f(3)$.

> $f(0) = 3$, $f(-1) = 6$, $f(3) = 18$
> $f(-1) = 2 + 1 + 3 = 6$ and $f(3) = 18 - 3 + 3 = 18$.

4. For $g(x) = x^2 + 4x$, find $g(a)$, $g(a+2)$ and $g(2x)$.

> $g(a) = a^2 + 4a$, $g(a+2) = a^2 + 8a + 12$, $g(2x) = 4x^2 + 8x$
> $g(a+2) = (a+2)^2 + 4(a+2) = a^2 + 4a + 4 + 4a + 8$. $g(2x) = (2x)^2 + 4(2x)$.

5. For $h(x) = 5 - 3x$, find the value of x for which (a) $h(x) = -4$ (b) $h(x) = 11$.

> **(a)** $x = 3$ **(b)** $x = -2$
> (a) $5 - 3x = -4$ gives $x = 3$. (b) $5 - 3x = 11$ gives $x = -2$.

Domain and range

6. Write down the domain and range of each function.

(a) $y = 2x - 7$ (b) $y = x^2 + 1$ (c) $y = -2x^2$ (d) $y = \dfrac{1}{x - 2}$

> **(a) all real x; all real y (b) all real x; $y \geq 1$ (c) all real x; $y \leq 0$ (d) $x \neq 2$; $y \neq 0$**
> (b) $x^2 \geq 0$, so $y \geq 1$. (c) $-2x^2 \leq 0$. (d) $x - 2 = 0$ is not allowed, and a fraction with numerator 1 is never 0.

7. Let $f(x) = x^2 - 6x + 5$.

(a) Find $f(0)$ and $f(5)$.

(b) Is f one-to-one? Explain.

(c) State the range of f.

> **(a) $f(0) = 5$, $f(5) = 0$. (b) No. (c) $y \geq -4$.**
> (b) $f(0) = 5$ and $f(6) = 5$, so two inputs give the same output. (c) $f(x) = (x - 3)^2 - 4$, so the smallest value is -4 at $x = 3$.

8. The function f is defined by $f(x) = ax^2 + b$, where $f(1) = 5$ and $f(-2) = 11$.

(a) Find a and b.

(b) State the range of f.

(c) Explain why f is not one-to-one.

> **(a) $a = 2$, $b = 3$. (b) $y \geq 3$. (c) $f(-1) = f(1)$.**
> (a) $a + b = 5$ and $4a + b = 11$. Subtract: $3a = 6$, so $a = 2$ and $b = 3$. (b) $2x^2 \geq 0$, so $f(x) \geq 3$. (c) $f(1) = 5$ and $f(-1) = 5$, so two inputs give the same output.