MCINTYRE MATHS · MATHEMATICS EXTENSION 2

Worked solutions

Lesson 9 · Further induction III: induction in context

Warm-up · Part A

(a) If $a_1 = 2$ and $a_{n+1} = 3a_n - 2$, find a_2, a_3, a_4.

> $a_2 = 4$, $a_3 = 10$, $a_4 = 28$. **A pattern:** $a_n = 3^{n-1} + 1$.
> Apply the rule repeatedly: $3(2) - 2 = 4$, $3(4) - 2 = 10$, $3(10) - 2 = 28$. These equal $3^1 + 1$, $3^2 + 1$, $3^3 + 1$.

(b) Differentiate xe^x, then differentiate the answer again.

> $(x+1)e^x$, **then** $(x+2)e^x$. **A pattern:** $(x+n)e^x$.
> Product rule: $e^x + xe^x = (x+1)e^x$. Again: $e^x + (x+1)e^x = (x+2)e^x$.

(c) Write $\sin 2A$ and $\cos 2A$ in terms of $\sin A$ and $\cos A$.

> $\sin 2A = 2\sin A \cos A$ **and** $\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$.
> These are the double angle formulas. The three forms of $\cos 2A$ come from $\sin^2 A + \cos^2 A = 1$.

(d) Show that $1 + \cos\theta = 2\cos^2\dfrac{\theta}{2}$.

> **Put** $A = \dfrac{\theta}{2}$ **in** $\cos 2A = 2\cos^2 A - 1$: $\cos\theta = 2\cos^2\dfrac{\theta}{2} - 1$, **so** $1 + \cos\theta = 2\cos^2\dfrac{\theta}{2}$.
> Parts (a), (b) and (d) are all used later in the lesson.

Example 1

Prove by induction that if $a_1 = 2$ and $a_{n+1} = 3a_n - 2$, then $a_n = 3^{n-1} + 1$ for all integers $n \geq 1$.

> **Let $P(n)$ be the statement $a_n = 3^{n-1} + 1$.**
> **Step 1.** $P(1)$: $3^0 + 1 = 2 = a_1$, **so $P(1)$ is true.**
> **Step 2. Assume $P(k)$: $a_k = 3^{k-1} + 1$ for some integer $k \geq 1$.**
> **Step 3.**
>
> $$a_{k+1} = 3a_k - 2 = 3\left(3^{k-1} + 1\right) - 2 = 3^k + 1.$$
>
> **So $P(k+1)$ is true when $P(k)$ is true.**
> **Step 4. True for $n = 1$, and true for $n = k \Rightarrow n = k+1$, so true for all integers $n \geq 1$ by induction.**
> The assumption is used inside the recursion: replace a_k in $a_{k+1} = 3a_k - 2$ by the assumed formula. This differs from series, where you add a term to the assumed sum.

Example 2

A sequence is defined by $u_1 = 1$ and $u_{n+1} = \dfrac{u_n}{1+u_n}$. Find u_2, u_3, u_4, make a conjecture for u_n, and prove it by induction.

$u_2 = \dfrac{1}{2}$, $u_3 = \dfrac{1}{3}$, $u_4 = \dfrac{1}{4}$. **Conjecture:** $u_n = \dfrac{1}{n}$.

Step 1. $u_1 = 1 = \dfrac{1}{1}$.

Step 2. Assume $u_k = \dfrac{1}{k}$ **for some integer** $k \geq 1$.

Step 3.

$$u_{k+1} = \frac{u_k}{1+u_k} = \frac{1/k}{1+1/k} = \frac{1/k}{(k+1)/k} = \frac{1}{k+1}.$$

Step 4. True for $n = 1$**, and true for** $n = k \Rightarrow n = k+1$**, so true for all integers** $n \geq 1$ **by induction.**

Conjecture first from the pattern, then prove it: this is the reasoning and communication the syllabus asks for. In Step 3, multiply top and bottom of the complex fraction by k.

HSC practice · A nested square root

The sequence a_n is defined by $a_1 = \sqrt{2}$ and $a_{n+1} = \sqrt{2 + a_n}$ for $n \geq 1$. Prove by induction that $a_n = 2\cos\left(\dfrac{\pi}{2^{n+1}}\right)$ for all integers $n \geq 1$. You may use $\sqrt{2+2\cos\theta} = 2\cos\dfrac{\theta}{2}$ for $0 \leq \theta \leq \pi$ (warm-up (d)).

Step 1. $P(1)$**:** $2\cos\dfrac{\pi}{4} = 2 \times \dfrac{\sqrt{2}}{2} = \sqrt{2} = a_1$**, so** $P(1)$ **is true.**

Step 2. Assume $a_k = 2\cos\dfrac{\pi}{2^{k+1}}$ **for some integer** $k \geq 1$.

Step 3.

$$a_{k+1} = \sqrt{2 + 2\cos\frac{\pi}{2^{k+1}}} = \sqrt{4\cos^2\frac{\pi}{2^{k+2}}} = 2\cos\frac{\pi}{2^{k+2}},$$

taking the positive root since $0 < \dfrac{\pi}{2^{k+2}} < \dfrac{\pi}{2}$**, so the cosine is positive.**

Step 4. True for $n = 1$**, and true for** $n = k \Rightarrow n = k+1$**, so true for all integers** $n \geq 1$ **by induction.**

Use the given identity with $\theta = \dfrac{\pi}{2^{k+1}}$, which lies in $[0, \pi]$. Then $\dfrac{\theta}{2} = \dfrac{\pi}{2^{k+2}}$. The positive square root must be justified: $\sqrt{4\cos^2\phi} = 2|\cos\phi| = 2\cos\phi$ only because $\cos\phi > 0$. (Wording adapted from the paper.)

Additional example · Second-order recursion

A sequence is defined by $a_1 = 1$, $a_2 = 5$ and $a_{n+2} = 5a_{n+1} - 6a_n$ for integers $n \geq 1$.

(a) Find a_3 and a_4. (1 mark)

(b) Prove by mathematical induction that $a_n = 3^n - 2^n$ for all integers $n \geq 1$. (3 marks)

(c) Hence, or otherwise, show that $a_{n+1} = 3a_n + 2^n$ for all integers $n \geq 1$. (2 marks)

(a) $a_3 = 5a_2 - 6a_1 = 5(5) - 6(1) = 19$ **and** $a_4 = 5a_3 - 6a_2 = 5(19) - 6(5) = 65$.

(b) Let $P(n)$ **be the statement** $a_n = 3^n - 2^n$.

Base cases. $3^1 - 2^1 = 1 = a_1$ **and** $3^2 - 2^2 = 5 = a_2$, **so** $P(1)$ **and** $P(2)$ **are true.**

Inductive step. Assume $P(k)$ **and** $P(k+1)$ **are true for some integer** $k \geq 1$, **that is,** $a_k = 3^k - 2^k$ **and** $a_{k+1} = 3^{k+1} - 2^{k+1}$. **We prove** $P(k+2)$, **that is,** $a_{k+2} = 3^{k+2} - 2^{k+2}$.

$$\begin{aligned} a_{k+2} &= 5a_{k+1} - 6a_k && \text{(definition of the sequence)} \\ &= 5\left(3^{k+1} - 2^{k+1}\right) - 6\left(3^k - 2^k\right) && \text{(by the assumption)} \\ &= 15 \cdot 3^k - 10 \cdot 2^k - 6 \cdot 3^k + 6 \cdot 2^k \\ &= 9 \cdot 3^k - 4 \cdot 2^k \\ &= 3^{k+2} - 2^{k+2}. \end{aligned}$$

So $P(k+2)$ **is true whenever** $P(k)$ **and** $P(k+1)$ **are true.**

Conclusion. $P(1)$ **and** $P(2)$ **are true, and** $P(k)$ **and** $P(k+1)$ **together imply** $P(k+2)$. **Hence, by mathematical induction,** $a_n = 3^n - 2^n$ **for all integers** $n \geq 1$.

(c) Using (b),

$$3a_n + 2^n = 3\left(3^n - 2^n\right) + 2^n = 3^{n+1} - 2 \cdot 2^n = 3^{n+1} - 2^{n+1} = a_{n+1}.$$

Why two base cases? Each term depends on the two terms before it, so the inductive step assumes two consecutive cases. Starting from $P(1)$ alone, the step could not reach $P(3)$, because it would also need $P(2)$. Checking both starts the chain: $P(1), P(2) \Rightarrow P(3)$, then $P(2), P(3) \Rightarrow P(4)$, and so on.

Common errors. Checking only $P(1)$; replacing a_{k+2} by $3^{k+2} - 2^{k+2}$ at the start, which assumes the result; leaving $9 \cdot 3^k - 4 \cdot 2^k$ without writing it as $3^{k+2} - 2^{k+2}$.

Example 3

(a) Prove by induction that $\dfrac{d^n}{dx^n}\left(xe^x\right) = (x+n)e^x$ for all integers $n \geq 1$.

Step 1. $P(1)$**:** $\dfrac{d}{dx}\left(xe^x\right) = e^x + xe^x = (x+1)e^x$.

Step 2. Assume $\dfrac{d^k}{dx^k}\left(xe^x\right) = (x+k)e^x$ **for some integer** $k \geq 1$.

Step 3.

$$\frac{d^{k+1}}{dx^{k+1}}\left(xe^x\right) = \frac{d}{dx}\left[(x+k)e^x\right] = e^x + (x+k)e^x = (x+k+1)e^x.$$

Step 4. True for $n = 1$, **and true for** $n = k \Rightarrow n = k+1$, **so true for all integers** $n \geq 1$ **by induction.**

The $(k+1)$th derivative is the derivative of the kth derivative. Use the product rule on $(x+k)e^x$, treating k as a constant.

(b) Prove by induction that, for $\sin x \neq 0$, $\cos x \cos 2x \cos 4x \cdots \cos\left(2^{n-1}x\right) = \dfrac{\sin\left(2^n x\right)}{2^n \sin x}$.

Step 1. $P(1)$: $\dfrac{\sin 2x}{2\sin x} = \dfrac{2\sin x \cos x}{2\sin x} = \cos x$.

Step 2. Assume the product up to $\cos\left(2^{k-1}x\right)$ equals $\dfrac{\sin\left(2^k x\right)}{2^k \sin x}$ for some integer $k \geq 1$.

Step 3. Multiply by $\cos\left(2^k x\right)$:

$$\frac{\sin(2^k x)\cos(2^k x)}{2^k \sin x} = \frac{\frac{1}{2}\sin\left(2^{k+1}x\right)}{2^k \sin x} = \frac{\sin\left(2^{k+1}x\right)}{2^{k+1}\sin x}.$$

Step 4. True for $n = 1$, and true for $n = k \Rightarrow n = k+1$, so true for all integers $n \geq 1$ by induction.

The $(k+1)$th product is the kth product times one new factor, $\cos(2^k x)$. The double angle formula with $A = 2^k x$ gives $\sin A \cos A = \frac{1}{2}\sin 2A$.

HSC practice · A trigonometric sum

(i) Show that $\cot x - \cot 2x = \csc 2x$.

(ii) Hence prove by induction that for all integers $n \geq 1$,

$$\sum_{r=1}^{n} \csc\left(2^r x\right) = \cot x - \cot\left(2^n x\right).$$

(i)

$$\cot x - \cot 2x = \frac{\cos x}{\sin x} - \frac{\cos 2x}{2\sin x \cos x} = \frac{2\cos^2 x - \cos 2x}{2\sin x \cos x} = \frac{1}{\sin 2x} = \csc 2x.$$

(ii) Step 1. $P(1)$: $\csc 2x = \cot x - \cot 2x$ **by (i).**

Step 2. Assume $\displaystyle\sum_{r=1}^{k} \csc\left(2^r x\right) = \cot x - \cot\left(2^k x\right)$ for some integer $k \geq 1$.

Step 3. Add $\csc\left(2^{k+1}x\right) = \cot\left(2^k x\right) - \cot\left(2^{k+1}x\right)$, which is (i) with x replaced by $2^k x$:

$$\sum_{r=1}^{k+1} \csc\left(2^r x\right) = \cot x - \cot\left(2^k x\right) + \cot\left(2^k x\right) - \cot\left(2^{k+1}x\right) = \cot x - \cot\left(2^{k+1}x\right).$$

Step 4. True for $n = 1$, and true for $n = k \Rightarrow n = k+1$, so true for all integers $n \geq 1$ by induction.

In (i), $\cos 2x = 2\cos^2 x - 1$ makes the numerator 1, and $2\sin x \cos x = \sin 2x$. The key idea in (ii) is "(i) with x replaced by $2^k x$", which students often miss. (Cosec is written csc here; both are fine. Wording adapted from the paper.)

Spot the flaw · What's missing?

What is missing from this proof, and why would a marker deduct for it?

Nothing written is false, but three things are missing.

1. The step $2 + 2\cos 2\theta = 4\cos^2\theta$ (from $\cos 2\theta = 2\cos^2\theta - 1$, with $\theta = \dfrac{\pi}{2^{k+1}}$) is not shown.

2. $\sqrt{4\cos^2\theta} = 2|\cos\theta|$, so we must justify $\cos\dfrac{\pi}{2^{k+1}} > 0$. This is true because $0 < \dfrac{\pi}{2^{k+1}} \leq \dfrac{\pi}{4} < \dfrac{\pi}{2}$ for $k \geq 1$.

3. There is no conclusion naming $P(k+1)$ and the start value: "$P(1)$ is true, and $P(k) \Rightarrow P(k+1)$, so $P(n)$ is true for all $n \geq 1$ by induction."

Compare the 2022 HSC question earlier in this lesson, which has the same structure with $a_1 = \sqrt{2}$. In an induction proof every step must be justified, in particular the choice of the positive square root, and the conclusion must be stated.

Warm-up · Part B

(a) Evaluate $\binom{5}{2} + \binom{5}{3}$ and $\binom{6}{3}$. What do you notice?

$10 + 10 = 20 = \binom{6}{3}$.

This is Pascal's identity: $\binom{5}{2} + \binom{5}{3} = \binom{6}{3}$.

(b) Expand $(1 + x)^4$ using Pascal's triangle.

$1 + 4x + 6x^2 + 4x^3 + x^4$.

The row of Pascal's triangle 1, 4, 6, 4, 1 gives the coefficients.

(c) Simplify $\ln\left(1 - \frac{1}{r^2}\right)$ as a single log of a product of two fractions.

$\ln\left[\frac{r-1}{r} \times \frac{r+1}{r}\right]$.

$1 - \frac{1}{r^2} = \frac{r^2 - 1}{r^2} = \frac{(r-1)(r+1)}{r^2}$, which is the product $\frac{r-1}{r} \times \frac{r+1}{r}$. Equivalently $\ln\frac{r-1}{r} + \ln\frac{r+1}{r}$.

(d) How many diagonals does a hexagon have?

9.

Each of the 6 vertices joins to 3 non-adjacent vertices, and each diagonal is counted twice: $\frac{6 \times 3}{2} = 9$.

Example 4

(a) Prove Pascal's identity using the factorial formula.

Using $\binom{n}{r} = \frac{n!}{r!(n-r)!}$**:**

$$\binom{k}{r-1} + \binom{k}{r} = \frac{k!}{(r-1)!(k-r+1)!} + \frac{k!}{r!(k-r)!} = \frac{k!\,[r + (k-r+1)]}{r!(k-r+1)!} = \frac{(k+1)!}{r!(k+1-r)!} = \binom{k+1}{r}.$$

Use the common denominator $r!(k-r+1)!$. The first fraction needs a factor r (since $r! = r\,(r-1)!$) and the second needs $(k-r+1)$ (since $(k-r+1)! = (k-r+1)(k-r)!$). The numerator becomes $k!(k+1)$.

(b) Prove the binomial theorem by induction.

Step 1. $P(1)$: $(1+x)^1 = 1+x = \binom{1}{0} + \binom{1}{1}x$.

Step 2. Assume $(1+x)^k = \sum_{r=0}^{k} \binom{k}{r} x^r$ **for some integer** $k \geq 1$.

Step 3.

$$(1+x)^{k+1} = (1+x)\sum_{r=0}^{k}\binom{k}{r}x^r = \sum_{r=0}^{k}\binom{k}{r}x^r + \sum_{r=0}^{k}\binom{k}{r}x^{r+1}.$$

The coefficient of x^r **for** $1 \leq r \leq k$ **is** $\binom{k}{r} + \binom{k}{r-1} = \binom{k+1}{r}$ **by (a). The constant term is** $1 = \binom{k+1}{0}$ **and the coefficient of** x^{k+1} **is** $1 = \binom{k+1}{k+1}$. **Hence** $(1+x)^{k+1} = \sum_{r=0}^{k+1}\binom{k+1}{r}x^r$.

Step 4. True for $n = 1$**, and true for** $n = k \Rightarrow n = k+1$**, so true for all integers** $n \geq 1$ **by induction.**

Write out a few terms rather than relying only on the sigma notation: the first sum gives the coefficients of x^r and the second sum, shifted along by one power, gives the coefficients of x^r from $\binom{k}{r-1}$. The binomial theorem was used in Extension 1; this is where it is proved.

Example 5

(a) Given $\ln(ab) = \ln a + \ln b$, prove by induction that for positive $a_1, \ldots, a_n$: $\ln(a_1 a_2 \cdots a_n) = \sum_{i=1}^{n} \ln a_i$.

Step 1. $P(1)$: $\ln a_1 = \ln a_1$.

Step 2. Assume $\ln(a_1 \cdots a_k) = \sum_{i=1}^{k} \ln a_i$ **for some integer** $k \geq 1$.

Step 3.

$$\ln(a_1 \cdots a_k a_{k+1}) = \ln(a_1 \cdots a_k) + \ln a_{k+1} = \sum_{i=1}^{k} \ln a_i + \ln a_{k+1} = \sum_{i=1}^{k+1} \ln a_i.$$

Step 4. True for $n = 1$**, and true for** $n = k \Rightarrow n = k+1$**, so true for all integers** $n \geq 1$ **by induction.**

The log law $\ln(ab) = \ln a + \ln b$ is applied with $a = a_1 \cdots a_k$ and $b = a_{k+1}$. Then the assumption replaces $\ln(a_1 \cdots a_k)$.

(b) Prove by induction that for all integers $n \geq 2$: $\sum_{r=2}^{n} \ln\left(1 - \frac{1}{r^2}\right) = \ln\left(\frac{n+1}{2n}\right)$.

Step 1. $P(2)$: $\ln\left(1 - \frac{1}{4}\right) = \ln\frac{3}{4} = \ln\frac{3}{2\times 2}$, **true.**

Step 2. Assume the result for some integer $k \geq 2$.

Step 3.

$$\sum_{r=2}^{k+1} \ln\left(1 - \frac{1}{r^2}\right) = \ln\frac{k+1}{2k} + \ln\left(1 - \frac{1}{(k+1)^2}\right) = \ln\left[\frac{k+1}{2k} \times \frac{k(k+2)}{(k+1)^2}\right] = \ln\frac{k+2}{2(k+1)}.$$

Step 4. True for $n = 2$**, and true for** $n = k \Rightarrow n = k+1$**, so true for all integers** $n \geq 2$ **by induction.**

The starting value is $n = 2$. Use $1 - \frac{1}{(k+1)^2} = \frac{(k+1)^2 - 1}{(k+1)^2} = \frac{k(k+2)}{(k+1)^2}$. A non-induction method is to telescope using warm-up (c).

Example 6

Prove by induction that a convex polygon with n sides has $\dfrac{n(n-3)}{2}$ diagonals, for all integers $n \geq 3$.

Step 1. $P(3)$**: a triangle has 0 diagonals and** $\dfrac{3\times 0}{2} = 0$.
Step 2. Assume a convex k**-gon has** $\dfrac{k(k-3)}{2}$ **diagonals, for some integer** $k \geq 3$.
Step 3. Add a vertex V **outside the edge** AB **to make a convex** $(k+1)$**-gon. The new diagonals are** V **joined to the** $k-2$ **vertices other than** A **and** B**, and the old side** AB **is now a diagonal:** $k-1$ **new diagonals.**

$$\frac{k(k-3)}{2} + k - 1 = \frac{k^2-k-2}{2} = \frac{(k+1)(k-2)}{2} = \frac{(k+1)\big((k+1)-3\big)}{2}.$$

Step 4. True for $n=3$**, and true for** $n=k \Rightarrow n=k+1$**, so true for all integers** $n \geq 3$ **by induction.**
In a geometry proof, Step 3 must say in words what the $(k+1)$th step adds. Draw a hexagon becoming a heptagon: V is adjacent only to A and B, so it makes diagonals to the other $k-2$ vertices, and AB stops being a side.

Exit ticket

A sequence is defined by $a_1 = 5$ and $a_{n+1} = 2a_n - 3$. Prove by induction that $a_n = 2^n + 3$ for all integers $n \geq 1$.

Step 1. $P(1)$**:** $2+3 = 5 = a_1$**, so** $P(1)$ **is true.**
Step 2. Assume $a_k = 2^k + 3$ **for some integer** $k \geq 1$.
Step 3. $a_{k+1} = 2a_k - 3 = 2\left(2^k+3\right) - 3 = 2^{k+1} + 3$**. So** $P(k+1)$ **is true when** $P(k)$ **is true.**
Step 4. True for $n=1$**, and true for** $n=k \Rightarrow n=k+1$**, so true for all integers** $n \geq 1$ **by induction.**
Substitute the assumption into the recursion. Do not use the formula for a_{k+1} in Step 3, as that is what you are proving. Check: $a_2 = 7 = 2^2 + 3$.

Practice worksheet

Recursive sequences

1. A sequence has $a_1 = 3$ and $a_{n+1} = 2a_n - 1$. Prove by induction that $a_n = 2^n + 1$ for all integers $n \geq 1$.

Step 1. $n=1$**:** $2+1 = 3 = a_1$.
Step 2. Assume $a_k = 2^k + 1$ **for some integer** $k \geq 1$.
Step 3. $a_{k+1} = 2a_k - 1 = 2(2^k+1) - 1 = 2^{k+1} + 1$.
Step 4. True for $n=1$**, and true for** $n=k \Rightarrow n=k+1$**, so true for all integers** $n \geq 1$ **by induction.**

2. A sequence has $u_1 = 1$ and $u_{n+1} = u_n + 2n + 1$. Prove by induction that $u_n = n^2$ for all integers $n \geq 1$.

Step 1. $n=1$**:** $1^2 = 1 = u_1$.
Step 2. Assume $u_k = k^2$ **for some integer** $k \geq 1$.
Step 3. $u_{k+1} = u_k + 2k + 1 = k^2 + 2k + 1 = (k+1)^2$.
Step 4. True for $n=1$**, and true for** $n=k \Rightarrow n=k+1$**, so true for all integers** $n \geq 1$ **by induction.**

Calculus

3. Prove by induction that $\dfrac{d^n}{dx^n}\left(xe^{-x}\right) = (-1)^n(x-n)e^{-x}$ for all integers $n \geq 1$.

Step 1. $n=1$**:** $\dfrac{d}{dx}(xe^{-x}) = e^{-x} - xe^{-x} = (1-x)e^{-x} = (-1)^1(x-1)e^{-x}$.
Step 2. Assume the k**th derivative is** $(-1)^k(x-k)e^{-x}$ **for some integer** $k \geq 1$.
Step 3. Differentiate: $(-1)^k\left[e^{-x} - (x-k)e^{-x}\right] = (-1)^k(1+k-x)e^{-x} = (-1)^{k+1}\big(x-(k+1)\big)e^{-x}$.
Step 4. True for $n=1$**, and true for** $n=k \Rightarrow n=k+1$**, so true for all integers** $n \geq 1$ **by induction.**

Binomial theorem and logarithms

4. Use Pascal's identity to prove by induction that $\displaystyle\sum_{r=0}^{n}\binom{n}{r}=2^n$ for all integers $n\geq 1$.

Step 1. $n=1$: $\binom{1}{0}+\binom{1}{1}=2=2^1$.

Step 2. Assume $\displaystyle\sum_{r=0}^{k}\binom{k}{r}=2^k$ **for some integer** $k\geq 1$.

Step 3.

$$\sum_{r=0}^{k+1}\binom{k+1}{r}=1+\sum_{r=1}^{k}\left[\binom{k}{r-1}+\binom{k}{r}\right]+1=\sum_{r=0}^{k}\binom{k}{r}+\sum_{r=0}^{k}\binom{k}{r}=2^k+2^k=2^{k+1}.$$

(**Check:** $\displaystyle\sum_{r=1}^{k}\binom{k}{r-1}=\sum_{r=0}^{k-1}\binom{k}{r}=2^k-1$ **and** $\displaystyle\sum_{r=1}^{k}\binom{k}{r}=2^k-1$, **so the total is** $1+(2^k-1)+(2^k-1)+1=2^{k+1}$.)

Step 4. True for $n=1$, **and true for** $n=k\Rightarrow n=k+1$, **so true for all integers** $n\geq 1$ **by induction.**

5. Prove by induction that $\displaystyle\sum_{r=2}^{n}\ln\left(1-\frac{1}{r}\right)=-\ln n$ for all integers $n\geq 2$.

Step 1. $n=2$: $\ln\dfrac{1}{2}=-\ln 2$.

Step 2. Assume the result for some integer $k\geq 2$.

Step 3.

$$\sum_{r=2}^{k+1}\ln\left(1-\frac{1}{r}\right)=-\ln k+\ln\frac{k}{k+1}=-\ln k+\ln k-\ln(k+1)=-\ln(k+1).$$

Step 4. True for $n=2$, **and true for** $n=k\Rightarrow n=k+1$, **so true for all integers** $n\geq 2$ **by induction.**

Geometry

6. Prove by induction that the interior angles of a convex polygon with n sides sum to $(n-2)\times 180^\circ$, for all integers $n\geq 3$.

Step 1. $n=3$: **a triangle has angle sum** $180^\circ=(3-2)\times 180^\circ$.

Step 2. Assume a convex k**-gon has angle sum** $(k-2)\times 180^\circ$, **for some integer** $k\geq 3$.

Step 3. In a convex $(k+1)$**-gon** $A_1A_2\cdots A_{k+1}$, **draw the diagonal** A_1A_k. **It splits the polygon into the** k**-gon** $A_1A_2\cdots A_k$ **and the triangle** $A_1A_kA_{k+1}$. **The angles of the** $(k+1)$**-gon are the angles of the** k**-gon plus the angles of the triangle, so the sum is** $(k-2)\times 180^\circ+180^\circ=\big((k+1)-2\big)\times 180^\circ$.

Step 4. True for $n=3$, **and true for** $n=k\Rightarrow n=k+1$, **so true for all integers** $n\geq 3$ **by induction.**

Adapted HSC-style

7. A sequence is defined by $a_1=1$ and $a_{n+1}=\sqrt{2a_n+3}$ for $n\geq 1$. (a) Prove by induction that $0<a_n<3$ for all integers $n\geq 1$. (b) Prove by induction that $a_{n+1}>a_n$ for all integers $n\geq 1$.

(a) Step 1. $a_1=1\in(0,3)$. **Step 2. Assume** $0<a_k<3$. **Step 3. Then** $3<2a_k+3<9$, **so** $\sqrt{3}<a_{k+1}<3$, **hence** $0<a_{k+1}<3$. **Step 4. By induction, true for all** $n\geq 1$.

(b) Step 1. $a_2=\sqrt{5}>1=a_1$. **Step 2. Assume** $a_{k+1}>a_k$ **for some integer** $k\geq 1$. **Step 3. Then** $2a_{k+1}+3>2a_k+3$, **and taking square roots (both sides positive),** $a_{k+2}=\sqrt{2a_{k+1}+3}>\sqrt{2a_k+3}=a_{k+1}$. **Step 4. By induction,** $a_{n+1}>a_n$ **for all** $n\geq 1$.