Exam practice 6 marks · about 11 minutes

A sequence is defined by

$$a_1 = 1, \qquad a_2 = 5, \qquad a_{n+2} = 5a_{n+1} - 6a_n \quad \text{for integers } n \geq 1.$$

(a) Find a_3 and a_4. **1**

(b) Prove by mathematical induction that $a_n = 3^n - 2^n$ for all integers $n \geq 1$. **3**

(c) Hence, or otherwise, show that $a_{n+1} = 3a_n + 2^n$ for all integers $n \geq 1$. **2**

(a) $a_3 = 5a_2 - 6a_1 = 5(5) - 6(1) = 19, \qquad a_4 = 5a_3 - 6a_2 = 5(19) - 6(5) = 65.$

(b) Let $P(n)$ be the statement $a_n = 3^n - 2^n$.

Base cases. $3^1 - 2^1 = 1 = a_1$ and $3^2 - 2^2 = 5 = a_2$, so $P(1)$ and $P(2)$ are true.

Inductive step. Assume $P(k)$ and $P(k+1)$ are true for some integer $k \geq 1$, that is,

$$a_k = 3^k - 2^k \quad \text{and} \quad a_{k+1} = 3^{k+1} - 2^{k+1}.$$

We prove $P(k+2)$, that is, $a_{k+2} = 3^{k+2} - 2^{k+2}$.

$$\begin{aligned} a_{k+2} &= 5a_{k+1} - 6a_k && \text{(definition of the sequence)} \\ &= 5\left(3^{k+1} - 2^{k+1}\right) - 6\left(3^k - 2^k\right) && \text{(by the assumption)} \\ &= 15 \cdot 3^k - 10 \cdot 2^k - 6 \cdot 3^k + 6 \cdot 2^k \\ &= 9 \cdot 3^k - 4 \cdot 2^k \\ &= 3^{k+2} - 2^{k+2}. \end{aligned}$$

So $P(k+2)$ is true whenever $P(k)$ and $P(k+1)$ are true.

Conclusion. $P(1)$ and $P(2)$ are true, and $P(k)$ and $P(k+1)$ together imply $P(k+2)$. Hence, by mathematical induction, $a_n = 3^n - 2^n$ for all integers $n \geq 1$.

(c) Using (b),

$$\begin{aligned} 3a_n + 2^n &= 3\left(3^n - 2^n\right) + 2^n = 3^{n+1} - 3 \cdot 2^n + 2^n \\ &= 3^{n+1} - 2 \cdot 2^n = 3^{n+1} - 2^{n+1} = a_{n+1}. \end{aligned}$$

Why two base cases? Each term depends on the *two* terms before it, so the inductive step assumes two consecutive cases. Starting from $P(1)$ alone, the step could not reach $P(3)$, because it would also need $P(2)$. Checking both $P(1)$ and $P(2)$ starts the chain: $P(1), P(2) \Rightarrow P(3)$, then $P(2), P(3) \Rightarrow P(4)$, and so on.

Common errors

- Checking only $P(1)$, or assuming only $P(k)$ and then using a_{k-1} without justification.
- Replacing a_{k+2} by $3^{k+2} - 2^{k+2}$ at the start, which assumes the result.
- Leaving $9 \cdot 3^k - 4 \cdot 2^k$ without writing it as $3^{k+2} - 2^{k+2}$.

Marking guide

(a)	Correct a_3 and a_4	1
(b)	Correct proof, including both base cases and a conclusion	3
	Correct inductive step, but only one base case checked	2
	Verifies both base cases, or states the two-term assumption and substitutes it correctly	1
(c)	Correct proof	2
	Substitutes the result of (b) into $3a_n + 2^n$, or equivalent progress	1