MCINTYRE MATHS · MATHEMATICS EXTENSION 2

# Worked solutions

Lesson 8 · Further induction II: inequalities

## Warm-up

**(a)** For which positive integers $n$ is $n^2 > 2n + 1$?

> $n \geq 3$.
> $n^2 > 2n + 1 \iff (n-1)^2 > 2$, which needs $n - 1 \geq 2$ for a positive integer, so $n \geq 3$. Check: $n = 2$ gives $4 > 5$, false; $n = 3$ gives $9 > 7$, true.

**(b)** Show that $2k^2 - (k+1)^2 = (k-1)^2 - 2$.

> **Both sides equal $k^2 - 2k - 1$.**
> LHS: $2k^2 - (k^2 + 2k + 1) = k^2 - 2k - 1$. RHS: $(k^2 - 2k + 1) - 2 = k^2 - 2k - 1$.

**(c)** If $A > B$ and $B \geq C$, what can you conclude?

> $A > C$.
> A strict inequality followed by a non-strict one still gives a strict result. This is the idea behind the chain layout used in this lesson.

**(d)** Test $2^n > n^2$ for $n = 1, 2, 3, 4, 5, 6$.

> **True for $n = 1$, false for $n = 2, 3, 4$, true for $n = 5, 6$. So claim $n \geq 5$.**
> $2 > 1$ yes; $4 = 4$ no; $8 < 9$ no; $16 = 16$ no; $32 > 25$ yes; $64 > 36$ yes. The claim holds from $n = 5$ on, so the first value of $n$ must be chosen with care.

## Example 1

Prove by induction that $2^n > n^2$ for all integers $n \geq 5$.

> **Let $P(n)$: $2^n > n^2$.**
> **Step 1.** $P(5)$: $2^5 = 32 > 25 = 5^2$. $\therefore$ $P(5)$ **is true.**
> **Step 2. Assume $2^k > k^2$ for some integer $k \geq 5$.**
> **Step 3. Need $2^{k+1} > (k+1)^2$.**
>
> $$2^{k+1} = 2 \cdot 2^k > 2k^2 \geq (k+1)^2.$$
>
> **The last step holds because $2k^2 - (k+1)^2 = (k-1)^2 - 2 \geq 16 - 2 > 0$ for $k \geq 5$. $\therefore$ $P(k+1)$ is true whenever $P(k)$ is true.**
> **Step 4. By mathematical induction, $2^n > n^2$ for all integers $n \geq 5$.**
> The assumption gives the first link. The "gap" $2k^2 \geq (k+1)^2$ is proved separately (Warm-up (b)), and it is where $k \geq 5$ is used. Without a large enough $k$ the gap fails, for example at $k = 2$.

## Example 2 · Factorials and powers

**(a)** Prove by induction that $n! > 2^n$ for all integers $n \geq 4$.

> **Step 1.** $n = 4$: $4! = 24 > 16 = 2^4$.
> **Step 2. Assume $k! > 2^k$ for some integer $k \geq 4$.**
> **Step 3. Need $(k+1)! > 2^{k+1}$.**
>
> $$(k+1)! = (k+1)\,k! > (k+1)2^k \geq 2 \cdot 2^k = 2^{k+1},$$
>
> **since $k + 1 \geq 2$.**
> **Step 4. True for $n = 4$ and true for $n = k \Rightarrow n = k + 1$, so true for all integers $n \geq 4$ by induction.**
> Write $(k+1)! = (k+1)k!$ so the assumption can be used. The factor $(k+1)$ is at least 2, which turns $2^k$ into $2^{k+1}$. Check: $n = 3$ gives $6 > 8$, false, which is why the base case is $n = 4$.

**(b)** Prove by induction that $3^n \geq 1 + 2n$ for all integers $n \geq 1$.

> **Step 1.** $n = 1$: $3^1 = 3 \geq 1 + 2 = 3$.
> **Step 2. Assume $3^k \geq 1 + 2k$ for some integer $k \geq 1$.**
> **Step 3. Need $3^{k+1} \geq 1 + 2(k+1) = 2k + 3$.**
>
> $$3^{k+1} = 3 \cdot 3^k \geq 3(1 + 2k) = 3 + 6k \geq 3 + 2k = 1 + 2(k+1),$$
>
> **since $4k \geq 0$.**
> **Step 4. True for $n = 1$ and true for $n = k \Rightarrow n = k + 1$, so true for all integers $n \geq 1$ by induction.**
> Multiply the assumption by 3 (positive, so the inequality direction is preserved). The gap is $3 + 6k - (3 + 2k) = 4k \geq 0$. The base case gives equality, so the inequality is $\geq$ rather than $>$.

## 2023 HSC Q13(b)

**(i)** Show that $k^2 - 2k - 3 \geq 0$ for all integers $k \geq 3$.

> $k^2 - 2k - 3 = (k-3)(k+1) \geq 0$ **for $k \geq 3$.**
> For $k \geq 3$ both factors are non-negative: $k - 3 \geq 0$ and $k + 1 > 0$.

**(ii)** Hence prove by induction that $2^n \geq n^2 - 2$ for all integers $n \geq 3$.

> **Step 1.** $P(3)$: $8 \geq 9 - 2 = 7$. **True.**
> **Step 2. Assume $2^k \geq k^2 - 2$ for some integer $k \geq 3$.**
> **Step 3. Then $2^{k+1} = 2 \cdot 2^k \geq 2k^2 - 4$. Need $2^{k+1} \geq (k+1)^2 - 2 = k^2 + 2k - 1$. Now**
>
> $$2k^2 - 4 - (k^2 + 2k - 1) = k^2 - 2k - 3 \geq 0$$
>
> **by (i), so $2^{k+1} \geq 2k^2 - 4 \geq (k+1)^2 - 2$.**
> **Step 4. True for $n = 3$ and true for $n = k \Rightarrow n = k + 1$, so true for all integers $n \geq 3$ by induction.**
> Part (i) is exactly the gap between $2k^2 - 4$ and the target $(k+1)^2 - 2$. Look for the given result inside the inductive step.

## Question A

Prove by induction that $2^n < \sqrt{n!}$ for all integers $n \geq 9$.

**Step 1.** $P(9)$**:** $2^9 = 512$ **and** $\sqrt{9!} = \sqrt{362\,880} \approx 602.4$**, so** $512 < 602.4$**. True.**
**Step 2. Assume** $2^k < \sqrt{k!}$ **for some integer** $k \geq 9$**.**
**Step 3.**

$$2^{k+1} = 2 \cdot 2^k < 2\sqrt{k!} < \sqrt{k+1}\,\sqrt{k!} = \sqrt{(k+1)!},$$

**since** $\sqrt{k+1} \geq \sqrt{10} > 2$**.**
**Step 4. True for** $n = 9$ **and true for** $n = k \Rightarrow n = k+1$**, so true for all integers** $n \geq 9$ **by induction.**
Split $\sqrt{(k+1)!} = \sqrt{k+1}\sqrt{k!}$ so the assumption can be used, then compare the factor $\sqrt{k+1}$ with 2. This is where $k \geq 9$ is needed. For the base case, $602^2 = 362\,404 < 9! = 362\,880$, so $\sqrt{9!} > 602 > 512$.

## Question B

Prove by induction that $2^n + 3^n < 5^n$ for all integers $n \geq 2$.

**Step 1.** $P(2)$**:** $4 + 9 = 13 < 25$**. True.**
**Step 2. Assume** $2^k + 3^k < 5^k$ **for some integer** $k \geq 2$**.**
**Step 3.**

$$2^{k+1} + 3^{k+1} = 2 \cdot 2^k + 3 \cdot 3^k < 3\left(2^k + 3^k\right) < 3 \cdot 5^k < 5^{k+1}.$$

**Step 4. True for** $n = 2$ **and true for** $n = k \Rightarrow n = k+1$**, so true for all integers** $n \geq 2$ **by induction.**
Replace both multipliers by the larger one, 3, so the bracket $2^k + 3^k$ from the assumption appears. Then $3 \cdot 5^k < 5 \cdot 5^k$. A non-induction proof expands $(2+3)^n$ by the binomial theorem (Lesson 3).

## HSC practice · A binomial coefficient

Prove by induction that $\binom{2n}{n} < 2^{2n-2}$ for all integers $n \geq 5$.

Hint: $\binom{2k+2}{k+1} = \dfrac{2(2k+1)}{k+1}\binom{2k}{k}$.

**Step 1.** $P(5)$: $\binom{10}{5} = 252 < 2^8 = 256$. **True.**

**Step 2. Assume** $\binom{2k}{k} < 2^{2k-2}$ **for some integer** $k \geq 5$.

**Step 3.**

$$\binom{2k+2}{k+1} = \frac{2(2k+1)}{k+1}\binom{2k}{k} < 4\binom{2k}{k} < 4 \cdot 2^{2k-2} = 2^{2(k+1)-2},$$

**since** $\dfrac{2k+1}{k+1} < 2$.

**Step 4. True for** $n = 5$ **and true for** $n = k \Rightarrow n = k+1$**, so true for all integers** $n \geq 5$ **by induction.**

The hint comes from $\binom{2k+2}{k+1} = \dfrac{(2k+2)(2k+1)}{(k+1)(k+1)}\binom{2k}{k}$ using the factorial formula; prove it first if it is not given. The ratio $\dfrac{2(2k+1)}{k+1} = 4 - \dfrac{2}{k+1}$ is less than 4, and $\binom{2k}{k} > 0$ keeps the inequality direction.

## HSC practice · A sum

Prove by induction that

$$\frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} < \frac{n-1}{n}$$

for all integers $n \geq 2$.

**Step 1.** $P(2)$: $\dfrac{1}{4} < \dfrac{1}{2}$. **True.**

**Step 2. Assume** $\displaystyle\sum_{r=2}^{k} \frac{1}{r^2} < \frac{k-1}{k}$ **for some integer** $k \geq 2$.

**Step 3.** $\displaystyle\sum_{r=2}^{k+1} \frac{1}{r^2} < \frac{k-1}{k} + \frac{1}{(k+1)^2}$**. Need this to be at most** $\dfrac{k}{k+1}$**. Now**

$$\frac{k}{k+1} - \frac{k-1}{k} = \frac{1}{k(k+1)} > \frac{1}{(k+1)^2},$$

**so** $\dfrac{k-1}{k} + \dfrac{1}{(k+1)^2} < \dfrac{k}{k+1}$.

**Step 4. True for** $n = 2$ **and true for** $n = k \Rightarrow n = k+1$**, so true for all integers** $n \geq 2$ **by induction.**

Add the next term to the assumption, then compare with the target by looking at the difference. Since $k(k+1) < (k+1)^2$, we have $\frac{1}{k(k+1)} > \frac{1}{(k+1)^2}$. Non-induction alternative: $\frac{1}{r^2} < \frac{1}{r(r-1)} = \frac{1}{r-1} - \frac{1}{r}$, which telescopes.

## Example 3 · A sum with square roots

Prove by induction that

$$\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\cdots+\frac{1}{\sqrt{n}}\geq\sqrt{n}$$

for all integers $n\geq 1$.

**Step 1.** $n=1$**: LHS** $=1\geq 1=\sqrt{1}$**.**
**Step 2. Assume** $S_k\geq\sqrt{k}$ **for some integer** $k\geq 1$**.**
**Step 3. Need** $S_{k+1}\geq\sqrt{k+1}$**.**

$$S_{k+1}=S_k+\frac{1}{\sqrt{k+1}}\geq\sqrt{k}+\frac{1}{\sqrt{k+1}}.$$

**Now** $\sqrt{k}+\dfrac{1}{\sqrt{k+1}}\geq\sqrt{k+1}$ **because, multiplying by** $\sqrt{k+1}>0$**, this is equivalent to** $\sqrt{k(k+1)}+1\geq k+1$**, that is** $\sqrt{k^2+k}\geq k$**, which is true.**
**Step 4. True for** $n=1$ **and true for** $n=k\Rightarrow n=k+1$**, so true for all integers** $n\geq 1$ **by induction.**
The gap inequality is a small separate problem. It is safe to multiply by $\sqrt{k+1}$ because it is positive, and every step is reversible, so the gap inequality really does follow from the true statement $\sqrt{k^2+k}\geq k$ (as $k^2+k\geq k^2$).

## Spot the flaw · An induction inequality

**Task** Find the flaw, explain why it matters, then write a correct version.

**Flaw: line 3 is not justified.** $2k^2-(k+1)^2=(k-1)^2-2$**, which is** $\geq 0$ **only for** $k\geq 3$ **(**$k=1$**:** $2<4$**;** $k=2$**:** $8<9$**). The claim is also false for** $n=2,3,4$ **(**$4=4$**,** $8<9$**,** $16=16$**).**
**Correct version: Example 1. Start at** $n=5$ **(**$32>25$**) and prove the gap step for** $k\geq 5$**.**
Prove the "gap" inequality for every $k$ the step needs, and test small cases before choosing the first value.

## Exit ticket

**(a)** Find the smallest positive integer $n$ for which it is true (and so the correct base case).

$n=3$**.**
$n=1$: $4>13$ false. $n=2$: $16>16$ false. $n=3$: $64>19$ true.

**(b)** Assume $4^k>3k+10$ for some integer $k\geq 3$. Write only the inductive step (Step 3) as a chain of inequalities.

$$4^{k+1}=4\cdot 4^k>4(3k+10)=12k+40>3k+13=3(k+1)+10.$$

The gap is $12k+40-(3k+13)=9k+27>0$. Each link is justified: the first by the assumption, the second because $9k+27>0$.

## Practice worksheet

### Choosing the first value

**1.** Find the smallest positive integer $n$ from which each inequality is true for all larger $n$ as well, by testing small values.

(a) $2^n > 2n + 1$ (b) $n! > n^2$

(c) $2^n > n^2$

> **(a)** $n = 3$ **(**$2 < 3, 4 < 5, 8 > 7$**). (b)** $n = 4$ **(**$1 = 1, 2 < 4, 6 < 9, 24 > 16$**). (c)** $n = 5$ **(**$2 > 1$ **but** $4 = 4, 8 < 9, 16 = 16$**, then** $32 > 25$**).**

### Induction with inequalities

**2.** Prove by induction that $2^n > 2n + 1$ for all integers $n \geq 3$.

> **Step 1.** $n = 3$**:** $8 > 7$ **true.**
> **Step 2. Assume** $2^k > 2k + 1$ **for some integer** $k \geq 3$**.**
> **Step 3.** $2^{k+1} = 2 \cdot 2^k > 2(2k + 1) = 4k + 2 \geq 2k + 3 = 2(k + 1) + 1$**, since** $2k \geq 1$**.**
> **Step 4. True for** $n = 3$**, and true for** $n = k \Rightarrow n = k + 1$**, so true for all integers** $n \geq 3$ **by induction.**

**3.** Prove by induction that $4^n \geq 1 + 3n$ for all integers $n \geq 1$.

> **Step 1.** $n = 1$**:** $4 \geq 4$ **true.**
> **Step 2. Assume** $4^k \geq 1 + 3k$ **for some integer** $k \geq 1$**.**
> **Step 3.** $4^{k+1} = 4 \cdot 4^k \geq 4(1 + 3k) = 4 + 12k \geq 4 + 3k = 1 + 3(k + 1)$**, since** $9k \geq 0$**.**
> **Step 4. True for** $n = 1$**, and true for** $n = k \Rightarrow n = k + 1$**, so true for all integers** $n \geq 1$ **by induction.**

**4.** Prove by induction that $n! \geq 2^{n-1}$ for all integers $n \geq 1$.

> **Step 1.** $n = 1$**:** $1! = 1 \geq 2^0 = 1$ **true.**
> **Step 2. Assume** $k! \geq 2^{k-1}$ **for some integer** $k \geq 1$**.**
> **Step 3.** $(k + 1)! = (k + 1)\,k! \geq (k + 1)2^{k-1} \geq 2 \cdot 2^{k-1} = 2^k = 2^{(k+1)-1}$**, since** $k + 1 \geq 2$**.**
> **Step 4. True for** $n = 1$**, and true for** $n = k \Rightarrow n = k + 1$**, so true for all integers** $n \geq 1$ **by induction.**

**5.** Prove by induction that $(1 + x)^n \geq 1 + nx$ for all integers $n \geq 1$, where $x > -1$ is a fixed real number.

> **Step 1.** $n = 1$**:** $1 + x \geq 1 + x$ **true.**
> **Step 2. Assume** $(1 + x)^k \geq 1 + kx$ **for some integer** $k \geq 1$**.**
> **Step 3. Since** $1 + x > 0$**,** $(1 + x)^{k+1} = (1 + x)(1 + x)^k \geq (1 + x)(1 + kx) = 1 + (k + 1)x + kx^2 \geq 1 + (k + 1)x$**, because** $kx^2 \geq 0$**.**
> **Step 4. True for** $n = 1$**, and true for** $n = k \Rightarrow n = k + 1$**, so true for all integers** $n \geq 1$ **by induction.**

**6.** Prove by induction that $\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n}\geq\frac{1}{2}$ for all integers $n\geq 1$.

**Step 1.** $n=1$**: LHS** $=\frac{1}{2}\geq\frac{1}{2}$ **true.**
**Step 2. Let** $T_k=\frac{1}{k+1}+\cdots+\frac{1}{2k}$ **and assume** $T_k\geq\frac{1}{2}$ **for some integer** $k\geq 1$**.**
**Step 3.** $T_{k+1}=T_k-\frac{1}{k+1}+\frac{1}{2k+1}+\frac{1}{2k+2}=T_k+\frac{1}{2k+1}-\frac{1}{2k+2}$**, because** $\frac{1}{k+1}=\frac{2}{2k+2}$**. Then** $\frac{1}{2k+1}-\frac{1}{2k+2}=\frac{1}{(2k+1)(2k+2)}>0$**, so** $T_{k+1}>T_k\geq\frac{1}{2}$**.**
**Step 4. True for** $n=1$**, and true for** $n=k\Rightarrow n=k+1$**, so true for all integers** $n\geq 1$ **by induction.**

**Adapted HSC-style**

**7.** (a) Show that $2k^3-3k^2-3k-1>0$ for integers $k\geq 4$. (b) Hence prove by induction that $3^n>n^3$ for all integers $n\geq 4$.

**(a)** $2k^3-3k^2-3k-1=k^2(2k-3)-3k-1\geq 5k^2-3k-1=k(5k-3)-1\geq 4\times 17-1>0$ **for** $k\geq 4$**. (b) Step 1.** $n=4$**:** $81>64$ **true.**
**Step 2. Assume** $3^k>k^3$ **for some integer** $k\geq 4$**.**
**Step 3.** $3^{k+1}=3\cdot 3^k>3k^3=(k+1)^3+(2k^3-3k^2-3k-1)>(k+1)^3$ **by part (a).**
**Step 4. True for** $n=4$**, and true for** $n=k\Rightarrow n=k+1$**, so true for all integers** $n\geq 4$ **by induction.**