MCINTYRE MATHS · MATHEMATICS EXTENSION 2

Worked solutions

Lesson 7 · Further induction I: series and divisibility

Warm-up · Algebra for induction

(a) Write out the first three terms and the $(k+1)$th term of $\sum_{r=1}^{n} r(r+1)$.

> $1\times 2 + 2\times 3 + 3\times 4 + \ldots$; **the** $(k+1)$**th term is** $(k+1)(k+2)$.
> Replace r by $1, 2, 3$, and by $k+1$ for the general next term.

(b) Simplify $\dfrac{k(k+1)(k+2)}{3} + (k+1)(k+2)$.

> $\dfrac{(k+1)(k+2)(k+3)}{3}$.
> Take out the common factor $(k+1)(k+2)$ rather than expanding: $(k+1)(k+2)\left[\frac{k}{3}+1\right] = (k+1)(k+2)\cdot\frac{k+3}{3}$.

(c) Given $3^k - 1 = 2M$, write $3^{k+1} - 1$ in terms of M.

> $3^{k+1} - 1 = 2(3M+1)$.
> $3^k = 2M+1$, so $3^{k+1} - 1 = 3(2M+1) - 1 = 6M + 2 = 2(3M+1)$.

(d) If k is odd, what is the next odd integer?

> $k+2$.
> Odd integers go up in steps of 2. This matters in Example 4.

Example 1 · Extension 1 recap

Prove by mathematical induction that for all integers $n \geq 1$,

$$\sum_{r=1}^{n} r(r+1) = \frac{n(n+1)(n+2)}{3}.$$

> **Let** $P(n)$ **be the statement** $\sum_{r=1}^{n} r(r+1) = \dfrac{n(n+1)(n+2)}{3}$.
> **Step 1.** $P(1)$**: LHS** $= 1\times 2 = 2$**, RHS** $= \dfrac{1\times 2\times 3}{3} = 2$. $\therefore$ $P(1)$ **is true.**
> **Step 2. Assume** $P(k)$**:** $\sum_{r=1}^{k} r(r+1) = \dfrac{k(k+1)(k+2)}{3}$ **for some integer** $k \geq 1$.
> **Step 3. Need to show** $P(k+1)$**:** $\sum_{r=1}^{k+1} r(r+1) = \dfrac{(k+1)(k+2)(k+3)}{3}$.
> **LHS** $= \dfrac{k(k+1)(k+2)}{3} + (k+1)(k+2)$ **(by the assumption)** $= \dfrac{(k+1)(k+2)(k+3)}{3}$ $=$ **RHS (Warm-up (b)).**
> $\therefore$ $P(k+1)$ **is true whenever** $P(k)$ **is true.**
> **Step 4. Hence, by mathematical induction,** $P(n)$ **is true for all integers** $n \geq 1$.
> Mark the use of the assumption, and factorise the common factor instead of expanding. Writing what you need to show first tells you the target for the algebra.

Example 2 · A series with fractions

Prove by induction that for all integers $n \geq 1$,

$$\frac{1}{1\times 3}+\frac{1}{3\times 5}+\cdots+\frac{1}{(2n-1)(2n+1)}=\frac{n}{2n+1}.$$

Step 1. $n=1$: **LHS** $=\dfrac{1}{1\times 3}=\dfrac{1}{3}$, **RHS** $=\dfrac{1}{3}$. $\therefore P(1)$ **is true.**

Step 2. Assume $S_k=\dfrac{k}{2k+1}$ **for some integer** $k\geq 1$.

Step 3. Need to show $S_{k+1}=\dfrac{k+1}{2k+3}$. **The** $(k+1)$**th term is** $\dfrac{1}{(2k+1)(2k+3)}$, **so by the assumption**

$$S_{k+1}=\frac{k}{2k+1}+\frac{1}{(2k+1)(2k+3)}=\frac{2k^2+3k+1}{(2k+1)(2k+3)}=\frac{(2k+1)(k+1)}{(2k+1)(2k+3)}=\frac{k+1}{2k+3}.$$

$\therefore P(k+1)$ **is true whenever** $P(k)$ **is true.**

Step 4. Hence, by mathematical induction, the statement is true for all integers $n\geq 1$.

The $(k+1)$th term is found by putting $n=k+1$ in $\frac{1}{(2n-1)(2n+1)}$, giving $\frac{1}{(2k+1)(2k+3)}$. Combining over the common denominator gives $k(2k+3)+1=2k^2+3k+1$, which factorises as $(2k+1)(k+1)$. The common factor $(2k+1)$ cancelling confirms the algebra is on track.

Example 3 · Divisibility

Prove by induction that $5^{2n}+3n-1$ is divisible by 9 for all integers $n\geq 1$.

Step 1. $P(1)$: $25+3-1=27=9\times 3$. $\therefore P(1)$ **is true.**

Step 2. Assume $5^{2k}+3k-1=9M$ **for some integer** M, **so** $5^{2k}=9M-3k+1$.

Step 3. Need to show $5^{2(k+1)}+3(k+1)-1$ **is divisible by 9.**

$$5^{2k+2}+3k+2=25\cdot 5^{2k}+3k+2=25(9M-3k+1)+3k+2=225M-72k+27=9(25M-8k+3),$$

which is divisible by 9 since $25M-8k+3$ **is an integer.** $\therefore P(k+1)$ **is true whenever** $P(k)$ **is true.**

Step 4. Hence, by mathematical induction, $5^{2n}+3n-1$ **is divisible by 9 for all integers** $n\geq 1$.

Rewrite $5^{2k+2}=25\cdot 5^{2k}$ and substitute 5^{2k} from the assumption. Every term must then contain a factor of 9. Note that $3(k+1)-1=3k+2$.

Example 4 · Stepping by 2

Prove by induction that 3^n+7^n is divisible by 10 for all odd positive integers n.

Step 1. $n=1$ **(the first odd positive integer):** $3+7=10$. $\therefore P(1)$ **is true.**

Step 2. Assume $3^k+7^k=10M$ **for some odd** k **and some integer** M.

Step 3. The next odd integer is $k+2$. **Need to show** $3^{k+2}+7^{k+2}$ **is divisible by 10.**

$$3^{k+2}+7^{k+2}=9\cdot 3^k+49\cdot 7^k=9(3^k+7^k)+40\cdot 7^k=90M+40\cdot 7^k=10\left(9M+4\cdot 7^k\right),$$

which is divisible by 10. $\therefore P(k+2)$ **is true whenever** $P(k)$ **is true.**

Step 4. Hence, by mathematical induction, 3^n+7^n **is divisible by 10 for all odd positive integers** n.

The common error is to prove the case $n=k+1$, but $k+1$ is even. Write $49\cdot 7^k=9\cdot 7^k+40\cdot 7^k$ so that $9(3^k+7^k)$ appears and the remainder $40\cdot 7^k$ is clearly a multiple of 10. The conclusion must say "for all odd positive integers n".

Example 5 · Two different powers

Prove by induction that $7^n - 3^n$ is divisible by 4 for all integers $n \geq 1$.

Step 1. $P(1)$**:** $7 - 3 = 4$**.** $\therefore P(1)$ **is true.**
Step 2. Assume $7^k - 3^k = 4M$ **for some integer** M**, so** $7^k = 4M + 3^k$**.**
Step 3. Need to show $7^{k+1} - 3^{k+1}$ **is divisible by 4.**

$$7^{k+1} - 3^{k+1} = 7(4M + 3^k) - 3 \cdot 3^k = 28M + 4 \cdot 3^k = 4\left(7M + 3^k\right),$$

which is divisible by 4. $\therefore P(k+1)$ **is true whenever** $P(k)$ **is true.**
Step 4. Hence, by mathematical induction, $7^n - 3^n$ **is divisible by 4 for all integers** $n \geq 1$**.**
Tip: replace the power with the larger base (7^k), then the smaller power collects into a multiple of the divisor: $7 \cdot 3^k - 3 \cdot 3^k = 4 \cdot 3^k$.

Exam-style · Induction, then an application

(i) Prove by induction that for all integers $n \geq 1$,

$$1 \times 1! + 2 \times 2! + \cdots + n \times n! = (n+1)! - 1.$$

Step 1. $n = 1$**: LHS** $= 1 \times 1! = 1$**, RHS** $= 2! - 1 = 1$**. True.**
Step 2. Assume true for $n = k$**:** $\displaystyle\sum_{r=1}^{k} r \times r! = (k+1)! - 1$**.**
Step 3. Need to show $\displaystyle\sum_{r=1}^{k+1} r \times r! = (k+2)! - 1$**. LHS** $= (k+1)! - 1 + (k+1) \times (k+1)!$ **(by the assumption)**
$= (k+1)!\,(1 + k + 1) - 1 = (k+2)(k+1)! - 1 = (k+2)! - 1 =$ **RHS.**
Step 4. Hence, by mathematical induction, the statement is true for all integers $n \geq 1$**.**
Factorise $(k+1)!$ from the first and last terms, then use $(k+2)(k+1)! = (k+2)!$.

(ii) Hence find the value of $1 \times 1! + 2 \times 2! + \cdots + 10 \times 10!$.

$11! - 1 = 39\,916\,799$**.**
Put $n = 10$ in the result from (i): $11! = 39\,916\,800$.

Spot the flaw · Induction

Task Find the flaw, explain why it matters, then write a correct version.

Flaw: there is no base case. $P(1)$ **gives** $1 + 1 + 1 = 3$**, which is odd, so the claim is false.**
Correct version: prove instead that $n^2 + n + 1$ **is odd, directly (since** $n(n+1)$ **is even,** $n(n+1) + 1$ **is odd) or by induction starting with** $P(1)$**: 3 is odd.**
The inductive step is valid ($P(k) \Rightarrow P(k+1)$), but without a true first case it proves nothing, as the chain of implications never starts. Both steps are needed, and the conclusion must name where the induction starts.

Exit ticket

(a) Prove by induction that $4^n + 14$ is divisible by 6 for all integers $n \geq 1$.

Let $P(n)$**:** $4^n + 14$ **is divisible by 6.**
Step 1. $n = 1$**:** $4 + 14 = 18 = 6 \times 3$**.** $\therefore P(1)$ **is true.**
Step 2. Assume $4^k + 14 = 6M$ **for some integer** M**, so** $4^k = 6M - 14$**.**
Step 3. $4^{k+1} + 14 = 4(6M - 14) + 14 = 24M - 42 = 6(4M - 7)$**, divisible by 6.** $\therefore P(k+1)$ **is true whenever** $P(k)$ **is true.**
Step 4. Hence, by mathematical induction, $P(n)$ **is true for all integers** $n \geq 1$**.**
Substitute for 4^k from the assumption and check every term has a factor of 6.

(b) A student proves that $3^n + 7^n$ is divisible by 10 for all odd n by assuming the result for $n = k$ and proving it for $n = k + 1$. State what is wrong.

> **For odd n, the next case after $n = k$ is $n = k + 2$, not $k + 1$.**
> $k + 1$ is even and is not in the set of cases being proved, so the induction never links one odd case to the next. The conclusion must also say "for all odd positive integers n".

Practice worksheet

Series

1. Prove by induction that for all integers $n \geq 1$, $1 \times 3 + 2 \times 4 + \cdots + n(n+2) = \dfrac{n(n+1)(2n+7)}{6}$.

> **Step 1.** $n = 1$: **LHS** $= 3$, **RHS** $= \dfrac{1 \times 2 \times 9}{6} = 3$. **True.**
> **Step 2. Assume** $\displaystyle\sum_{r=1}^{k} r(r+2) = \frac{k(k+1)(2k+7)}{6}$.
> **Step 3.** $\displaystyle\sum_{r=1}^{k+1} r(r+2) = \frac{k(k+1)(2k+7)}{6} + (k+1)(k+3) = \frac{(k+1)\left[k(2k+7) + 6(k+3)\right]}{6} = \frac{(k+1)(2k^2+13k+18)}{6} = \frac{(k+1)(k+2)(2k+9)}{6}$, **which is the required result since** $2(k+1)+7 = 2k+9$.
> **Step 4. Hence, by mathematical induction, true for all integers** $n \geq 1$.

2. Prove by induction that for all integers $n \geq 1$, $\displaystyle\sum_{r=1}^{n} r \cdot 2^r = (n-1)2^{n+1} + 2$.

> **Step 1.** $n = 1$: **LHS** $= 1 \cdot 2 = 2$, **RHS** $= 0 \cdot 4 + 2 = 2$. **True.**
> **Step 2. Assume** $\displaystyle\sum_{r=1}^{k} r \cdot 2^r = (k-1)2^{k+1} + 2$.
> **Step 3.** $\displaystyle\sum_{r=1}^{k+1} r \cdot 2^r = (k-1)2^{k+1} + 2 + (k+1)2^{k+1} = 2k \cdot 2^{k+1} + 2 = k \cdot 2^{k+2} + 2 = \big((k+1)-1\big)2^{(k+1)+1} + 2$.
> **Step 4. Hence, by mathematical induction, true for all integers** $n \geq 1$.

Divisibility

3. Prove by induction that $9^n + 7$ is divisible by 8 for all integers $n \geq 1$.

> **Step 1.** $n = 1$: $9 + 7 = 16 = 8 \times 2$. **True.**
> **Step 2. Assume** $9^k + 7 = 8M$, **so** $9^k = 8M - 7$.
> **Step 3.** $9^{k+1} + 7 = 9(8M - 7) + 7 = 72M - 56 = 8(9M - 7)$, **divisible by 8.**
> **Step 4. Hence, by mathematical induction, true for all integers** $n \geq 1$.

4. Prove by induction that $5^n - 4n - 1$ is divisible by 16 for all integers $n \geq 1$.

> **Step 1.** $n = 1$: $5 - 4 - 1 = 0 = 16 \times 0$. **True.**
> **Step 2. Assume** $5^k - 4k - 1 = 16M$, **so** $5^k = 16M + 4k + 1$.
> **Step 3.** $5^{k+1} - 4(k+1) - 1 = 5(16M + 4k + 1) - 4k - 5 = 80M + 16k = 16(5M + k)$, **divisible by 16.**
> **Step 4. Hence, by mathematical induction, true for all integers** $n \geq 1$.

5. Prove by induction that $3^n + 5^n$ is divisible by 8 for all odd positive integers n.

Step 1. $n = 1$: $3 + 5 = 8$. **True.**
Step 2. Assume $3^k + 5^k = 8M$ **for some odd** k.
Step 3. The next odd integer is $k + 2$: $3^{k+2} + 5^{k+2} = 9 \cdot 3^k + 25 \cdot 5^k = 9(3^k + 5^k) + 16 \cdot 5^k = 72M + 16 \cdot 5^k = 8(9M + 2 \cdot 5^k)$, **divisible by 8.**
Step 4. Hence, by mathematical induction, true for all odd positive integers n.

Adapted HSC-style

6. (a) Prove by induction that for all integers $n \geq 1$, $\displaystyle\sum_{r=1}^{n} \frac{r}{(r+1)!} = 1 - \frac{1}{(n+1)!}$. [3] (b) Hence find $\displaystyle\sum_{r=1}^{9} \frac{r}{(r+1)!}$ as a fraction. [1]

(a) Step 1. $n = 1$: **LHS** $= \dfrac{1}{2!} = \dfrac{1}{2}$, **RHS** $= 1 - \dfrac{1}{2!} = \dfrac{1}{2}$. **True.**
Step 2. Assume $\displaystyle\sum_{r=1}^{k} \frac{r}{(r+1)!} = 1 - \frac{1}{(k+1)!}$.
Step 3. $\displaystyle\sum_{r=1}^{k+1} \frac{r}{(r+1)!} = 1 - \frac{1}{(k+1)!} + \frac{k+1}{(k+2)!} = 1 - \frac{(k+2)-(k+1)}{(k+2)!} = 1 - \frac{1}{(k+2)!}$, **as required.**
Step 4. Hence, by mathematical induction, true for all integers $n \geq 1$.
(b) $n = 9$: $1 - \dfrac{1}{10!} = 1 - \dfrac{1}{3\,628\,800} = \dfrac{3\,628\,799}{3\,628\,800}$.