MCINTYRE MATHS · MATHEMATICS EXTENSION 2

Worked solutions

Lesson 6 · Algebraic inequalities II

Warm-up

(a) Prove $p^2 + q^2 \geq 2pq$.

> $p^2 + q^2 - 2pq = (p-q)^2 \geq 0$.
> Subtract the right-hand side and recognise a perfect square.

(b) Expand $\left(\sqrt{a} - \sqrt{b}\right)^2$ for $a, b \geq 0$.

> $a - 2\sqrt{ab} + b$.
> $(\sqrt{a})^2 = a$, $(\sqrt{b})^2 = b$ and $\sqrt{a}\sqrt{b} = \sqrt{ab}$ since $a, b \geq 0$. This will give AM–GM.

(c) Compare $|-3+5|$ with $|-3| + |5|$.

> $|-3+5| = 2$ **and** $|-3| + |5| = 8$**, so** $2 < 8$.
> The two sides are equal only when the numbers have the same sign. This is the idea of the triangle inequality.

(d) Can a triangle have sides 3 cm, 4 cm and 8 cm?

> **No.**
> $3 + 4 = 7 < 8$: the two shorter sides together are too short to reach across the longest side.

Example 1 (a)

Prove the AM–GM inequality.

> $\left(\sqrt{a} - \sqrt{b}\right)^2 \geq 0 \Rightarrow a + b - 2\sqrt{ab} \geq 0 \Rightarrow \dfrac{a+b}{2} \geq \sqrt{ab}$**. Equality holds if and only if** $\sqrt{a} = \sqrt{b}$**, that is** $a = b$**.**
> This is the expansion from the warm-up. A square is zero only when its base is zero. (A geometric proof also works: in a semicircle of diameter $a + b$, the half-chord at the join has length $\sqrt{ab}$ and cannot exceed the radius $\frac{a+b}{2}$.)

Example 1 (b)

Prove that for $x > 0$, $x + \dfrac{9}{x} \geq 6$. When is there equality?

> $x + \dfrac{9}{x} \geq 2\sqrt{x \cdot \dfrac{9}{x}} = 2\sqrt{9} = 6$**. Equality holds when** $x = \dfrac{9}{x}$**, so** $x^2 = 9$ **and** $x = 3$**.**
> Apply AM–GM to the two positive numbers x and $\dfrac{9}{x}$. Their product is the constant 9, which is what makes the bound useful.

HSC practice · Substituting into AM–GM

(i) Show that $a + b \geq 2\sqrt{ab}$ for all $a, b \geq 0$.
(ii) Hence show that $p^2 + 4q^2 \geq 4pq$ for all real p and q.

> **(i)** $\left(\sqrt{a} - \sqrt{b}\right)^2 \geq 0$, **so** $a + b \geq 2\sqrt{ab}$. **(ii) Put** $a = p^2$, $b = 4q^2$ **(both** ≥ 0**):** $p^2 + 4q^2 \geq 2\sqrt{4p^2q^2} = 4|pq| \geq 4pq$.
> Substitution is the key skill: choose a and b so that the square roots simplify. Check: $(p - 2q)^2 \geq 0$ gives the same result directly. (The original part (i) is proved geometrically from a diagram; the wording here is adapted.)

Example 2 · Using AM–GM

(a) Prove that $(a + b)\left(\frac{1}{a} + \frac{1}{b}\right) \geq 4$ for $a, b > 0$.

> $(a + b)\left(\frac{1}{a} + \frac{1}{b}\right) = 2 + \frac{a}{b} + \frac{b}{a} \geq 2 + 2 = 4$.
> Expand, then use $\frac{a}{b} + \frac{b}{a} \geq 2$ from Lesson 5. Alternatively apply AM–GM to each bracket: $a + b \geq 2\sqrt{ab}$ and $\frac{1}{a} + \frac{1}{b} \geq \frac{2}{\sqrt{ab}}$, then multiply.

(b) Prove that $(a + b)(b + c)(c + a) \geq 8abc$ for $a, b, c > 0$.

> $a + b \geq 2\sqrt{ab}$, $b + c \geq 2\sqrt{bc}$, $c + a \geq 2\sqrt{ca}$. **All sides are positive, so multiply:** $(a + b)(b + c)(c + a) \geq 8\sqrt{a^2b^2c^2} = 8abc$.
> Multiplying inequalities is valid when all the quantities are positive. The three square roots multiply to $\sqrt{(abc)^2} = abc$.

HSC practice · Building on AM–GM

(i) Using $\sqrt{xy} \leq \frac{x + y}{2}$, show that $\sqrt{abc} \leq \frac{a^2 + b^2 + 2c}{4}$.
(ii) Hence show that $\sqrt{abc} \leq \frac{a^2 + b^2 + c^2 + a + b + c}{6}$.

> **(i)** $\sqrt{abc} = \sqrt{ab \cdot c} \leq \frac{ab + c}{2}$, **and** $ab \leq \frac{a^2 + b^2}{2}$, **so** $\sqrt{abc} \leq \frac{\frac{a^2 + b^2}{2} + c}{2} = \frac{a^2 + b^2 + 2c}{4}$. **(ii) By symmetry,** $\sqrt{abc} \leq \frac{b^2 + c^2 + 2a}{4}$ **and** $\sqrt{abc} \leq \frac{c^2 + a^2 + 2b}{4}$. **Add the three:** $3\sqrt{abc} \leq \frac{2(a^2 + b^2 + c^2) + 2(a + b + c)}{4}$. **Divide by 3.**
> In (i) apply AM–GM with $x = ab$, $y = c$, then use $ab \leq \frac{a^2 + b^2}{2}$ (Lesson 5). In (ii) the left side is the same in every case, which is why adding the three symmetric versions works.

HSC practice · A condition and AM–GM

Let a, b, c be positive real numbers with $\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}=1$. Prove that $a\sqrt{bc}+b\sqrt{ca}+c\sqrt{ab}\le abc$.

AM–GM: $a\sqrt{bc}\le\dfrac{a(b+c)}{2}$, $b\sqrt{ca}\le\dfrac{b(c+a)}{2}$, $c\sqrt{ab}\le\dfrac{c(a+b)}{2}$. Add: $\text{LHS}\le\dfrac{2ab+2bc+2ca}{2}=ab+bc+ca$. Multiplying the condition by abc gives $bc+ca+ab=abc$. So $\text{LHS}\le abc$.

Multiply each AM–GM bound by the positive factor (a, b or c). Note that $\sqrt{bc}\le\frac{b+c}{2}$, so $a\sqrt{bc}\le\frac{a(b+c)}{2}$. The condition is used by clearing the fractions. (Wording adapted from the paper.)

Example 3

Prove the triangle inequality by showing $(|a|+|b|)^2-|a+b|^2\ge 0$.

$(|a|+|b|)^2-|a+b|^2=a^2+2|a||b|+b^2-(a^2+2ab+b^2)=2(|ab|-ab)\ge 0$. Both $|a|+|b|$ and $|a+b|$ are non-negative, so $|a+b|\le|a|+|b|$.

Since $ab\le|ab|$, the bracket $|ab|-ab$ is non-negative. Squares of non-negative numbers compare in the same order as the numbers themselves. The same inequality holds later for complex numbers and for vectors.

Example 4 · Using the triangle inequality

(a) Prove that $|a+b+c|\le|a|+|b|+|c|$.

$|a+b+c|\le|a+b|+|c|\le|a|+|b|+|c|$.

Apply the triangle inequality twice: first to $(a+b)$ and c, then to a and b.

(b) Prove that $|a-b|\ge|a|-|b|$. (Hint: write $a=(a-b)+b$.)

$|a|=|(a-b)+b|\le|a-b|+|b|$, so $|a-b|\ge|a|-|b|$.

Apply the triangle inequality to $(a-b)$ and b, then subtract $|b|$ from both sides.

Example 5 · Inequalities in geometry

(a) A rectangle has perimeter P. Prove that its area is at most $\dfrac{P^2}{16}$. When is there equality?

Let the sides be x, y with $2(x+y)=P$. By AM–GM, $xy\le\left(\dfrac{x+y}{2}\right)^2=\left(\dfrac{P}{4}\right)^2=\dfrac{P^2}{16}$. Equality holds when $x=y$, that is a square.

Use $\sqrt{xy}\le\frac{x+y}{2}$ and square both sides (both are non-negative). The semi-perimeter $x+y=\frac{P}{2}$ is fixed, so $\frac{x+y}{2}=\frac{P}{4}$.

(b) A right-angled triangle has legs a, b and hypotenuse c. Prove that $a+b\le c\sqrt{2}$.

$(a+b)^2=a^2+b^2+2ab\le 2(a^2+b^2)=2c^2$. Since $a+b>0$, $a+b\le c\sqrt{2}$.

Use $2ab\le a^2+b^2$ and Pythagoras, $a^2+b^2=c^2$. Taking the square root is valid because $a+b$ and $c\sqrt{2}$ are both positive.

(c) Two sides of a triangle are 5 cm and 9 cm. Find the possible lengths of the third side.

> $4 < x < 14$.
> The triangle inequality must hold for each side: $x < 5 + 9 = 14$, $9 < 5 + x$ gives $x > 4$ and $5 < 9 + x$ is always true. So $9 - 5 < x < 9 + 5$.

Extension · AM–GM for four and three numbers

(i) Using AM–GM for two numbers twice, show that for positive reals $\dfrac{a+b+c+d}{4} \geq \sqrt[4]{abcd}$.

> $\dfrac{a+b+c+d}{4} = \dfrac{1}{2}\left[\dfrac{a+b}{2} + \dfrac{c+d}{2}\right] \geq \dfrac{1}{2}\left[\sqrt{ab} + \sqrt{cd}\right] \geq \sqrt{\sqrt{ab}\sqrt{cd}} = \sqrt[4]{abcd}$.
> The first step uses AM–GM on each pair. The second step uses AM–GM again on the two numbers $\sqrt{ab}$ and $\sqrt{cd}$.

(ii) By choosing $d = \dfrac{a+b+c}{3}$ in (i), deduce that $\dfrac{a+b+c}{3} \geq \sqrt[3]{abc}$.

> **Let $m = \dfrac{a+b+c}{3}$, so $a+b+c = 3m$. Then (i) gives $\dfrac{3m+m}{4} = m \geq \sqrt[4]{abcm}$. So $m^4 \geq abcm$, hence $m^3 \geq abc$ (as $m > 0$), and $m \geq \sqrt[3]{abc}$.**
> Putting $d = m$ makes the left side of (i) equal to m itself. Raise both sides to the fourth power and divide by $m > 0$.

Exit ticket

(a) Prove that $x + \dfrac{4}{x} \geq 4$ for $x > 0$.

> **By AM–GM, $x + \dfrac{4}{x} \geq 2\sqrt{x \cdot \dfrac{4}{x}} = 2\sqrt{4} = 4$.**
> The product of the two terms is the constant 4. Equality holds at $x = 2$.

(b) Two sides of a triangle are 6 and 10. How many integer lengths are possible for the third side?

> **11.**
> $10 - 6 < x < 10 + 6$, so $4 < x < 16$. The integers 5 to 15 give 11 values.

Spot the flaw · AM–GM

A student wrote this. Find the flaw, explain why it matters, then write a correct version.

Claim. $x + \dfrac{1}{x} \geq 2$ for all real $x \neq 0$.

1. By AM–GM, $x + \dfrac{1}{x} \geq 2\sqrt{x \cdot \dfrac{1}{x}}$

2. $= 2\sqrt{1} = 2$. $\square$

> **The flaw: AM–GM holds only for non-negative numbers, and x may be negative. The claim is false: $x = -1$ gives $x + \dfrac{1}{x} = -2 < 2$. Correct version: for $x > 0$, $x + \dfrac{1}{x} \geq 2\sqrt{x \cdot \dfrac{1}{x}} = 2$, with equality at $x = 1$. (For $x < 0$, $x + \dfrac{1}{x} \leq -2$.)**
>
> For $x < 0$ the product $x \cdot \frac{1}{x} = 1$ is still defined, which hides the problem, but the two terms are negative so AM–GM does not apply. Check the conditions of a theorem before using it, the same habit as dividing only by a positive quantity.

Which method? · Proofs so far

(a) There are no integers m, n with $6m + 9n = 100$.

> **Contradiction (or a direct divisibility argument). First line: suppose $6m + 9n = 100$ for some integers m, n.**
>
> $6m + 9n = 3(2m + 3n)$ is divisible by 3, but 100 is not, which is a contradiction. The words "there are no" point to contradiction.

(b) For integers n, if $n^2 + 2n$ is odd, then n is odd.

> **Contrapositive. First line: we prove that if n is even, then $n^2 + 2n$ is even.**
>
> Let $n = 2k$. Then $n^2 + 2n = 4k^2 + 4k = 2(2k^2 + 2k)$, which is even. "If … odd, then … odd" points to the contrapositive.

(c) $\dfrac{a+b}{2} \geq \dfrac{2ab}{a+b}$ for $a, b > 0$.

> **Difference and squares. First line: $\text{LHS} - \text{RHS} = \dfrac{(a+b)^2 - 4ab}{2(a+b)}$.**
>
> This equals $\dfrac{(a-b)^2}{2(a+b)} \geq 0$, since the numerator is a square and $a + b > 0$.

(d) $n^2 - n$ is even for every integer n.

> **Direct proof (or cases). First line: $n^2 - n = n(n-1)$.**
>
> This is a product of consecutive integers, so one factor is even and the product is even. Alternatively use cases $n = 2k$ and $n = 2k + 1$.

(e) $\sqrt{5}$ is irrational.

> **Contradiction. First line: suppose $\sqrt{5} = \dfrac{p}{q}$ with p, q integers having no common factor.**
>
> Then $p^2 = 5q^2$, so $5 \mid p^2$, which gives $5 \mid p$ (lemma: prove it by cases $p = 5k + r$ with $r = 1, 2, 3, 4$, or by prime factorisation). Writing $p = 5k$ gives $5 \mid q$, contradicting no common factor.

(f) $x + \dfrac{16}{x} \geq 8$ for $x > 0$.

> **AM–GM. First line:** $x + \dfrac{16}{x} \geq 2\sqrt{x \cdot \dfrac{16}{x}}$.
> This equals $2\sqrt{16} = 8$, with equality at $x = 4$. A sum of positive terms with a fixed product points to AM–GM.

Practice worksheet

AM–GM

1. Prove that $x + \dfrac{16}{x} \geq 8$ for $x > 0$, and state when equality holds.

> **By AM–GM,** $x + \dfrac{16}{x} \geq 2\sqrt{x \cdot \dfrac{16}{x}} = 2\sqrt{16} = 8$**. Equality holds when** $x = \dfrac{16}{x}$**, so** $x = 4$**.**

2. Prove that $(a+1)(b+1)(a+b) \geq 8ab$ for $a, b > 0$.

> **By AM–GM,** $a + 1 \geq 2\sqrt{a}$**,** $b + 1 \geq 2\sqrt{b}$ **and** $a + b \geq 2\sqrt{ab}$**. All are positive, so multiplying:** $(a+1)(b+1)(a+b) \geq 8\sqrt{a}\sqrt{b}\sqrt{ab} = 8ab$**.**

3. Prove that $|a - b| \leq |a| + |b|$ for real a, b.

> $|a - b| = |a + (-b)| \leq |a| + |-b| = |a| + |b|$**, by the triangle inequality.**

4. Prove that $\big||a| - |b|\big| \leq |a - b|$. (Hint: use $|a| = |(a-b) + b|$ and $|b| = |(b-a) + a|$.)

> $|a| \leq |a - b| + |b|$ **gives** $|a| - |b| \leq |a - b|$**. Also** $|b| \leq |b - a| + |a| = |a - b| + |a|$ **gives** $|b| - |a| \leq |a - b|$**. So** $-|a - b| \leq |a| - |b| \leq |a - b|$**, which means** $\big||a| - |b|\big| \leq |a - b|$**.**

5. Two sides of a triangle are 7 cm and 12 cm. (a) Find the possible lengths of the third side. (b) How many integer lengths are possible? (c) If the triangle is isosceles, find its sides.

> **(a)** $12 - 7 < x < 12 + 7$**, so** $5 < x < 19$**. (b) The integers 6 to 18: 13 values. (c)** $7, 7, 12$ **(valid since** $14 > 12$**) or** $7, 12, 12$ **(valid).**

Geometry

6. A rectangle has area $36\ \text{cm}^2$. Prove that its perimeter is at least 24 cm, and state when it equals 24 cm.

> **Let the sides be** x, y **with** $xy = 36$**. By AM–GM,** $\dfrac{x+y}{2} \geq \sqrt{xy} = 6$**, so** $x + y \geq 12$ **and the perimeter** $2(x+y) \geq 24$**. Equality holds when** $x = y = 6$ **(a square).**

7. A right-angled triangle has hypotenuse 10 cm. Prove that its perimeter is at most $10 + 10\sqrt{2}$ cm. When is this achieved?

> **With legs** a, b**,** $(a+b)^2 = a^2 + b^2 + 2ab \leq 2(a^2 + b^2) = 200$**, so** $a + b \leq 10\sqrt{2}$ **(positive). Perimeter** $= a + b + 10 \leq 10 + 10\sqrt{2}$**. Equality holds when** $a = b = 5\sqrt{2}$ **(isosceles right-angled).**

Adapted HSC-style

8. (a) Prove that $(a+b)\left(\frac{1}{a}+\frac{1}{b}\right) \geq 4$ for $a, b > 0$. [2] (b) Hence show that if $a+b=1$, then $\frac{1}{a}+\frac{1}{b} \geq 4$. [1] (c) Hence show that if $a, b > 0$ and $a+b=1$, then $\frac{1}{a^2}+\frac{1}{b^2} \geq 8$. [2]

(a) $(a+b)\left(\frac{1}{a}+\frac{1}{b}\right) = 2+\frac{a}{b}+\frac{b}{a}$**, and** $\frac{a}{b}+\frac{b}{a}-2 = \frac{(a-b)^2}{ab} \geq 0$**, so the product is at least 4.**

(b) Put $a+b=1$ **in (a):** $1 \times \left(\frac{1}{a}+\frac{1}{b}\right) \geq 4$**. (c) Using** $x^2+y^2 \geq 2xy$ **with** $x=\frac{1}{a}$, $y=\frac{1}{b}$**:** $\frac{1}{a^2}+\frac{1}{b^2} \geq \frac{2}{ab} = 2\left(\frac{1}{a}+\frac{1}{b}\right)$**, because** $\frac{1}{a}+\frac{1}{b} = \frac{a+b}{ab} = \frac{1}{ab}$**. By (b) this is at least** $2 \times 4 = 8$**.**