MCINTYRE MATHS · MATHEMATICS EXTENSION 2

Worked solutions

Lesson 5 · Algebraic inequalities I

Warm-up · Inequality instincts

(a) Solve $3 - 2x > 7$.

> $x < -2$.
> Subtract 3 to get $-2x > 4$, then divide by -2. Dividing by a negative number reverses the inequality.

(b) Complete the square: $x^2 - 6x + 11$.

> **$(x-3)^2 + 2$, so the expression is always at least 2.**
> $x^2 - 6x + 9 = (x-3)^2$, leaving $+2$. A square is never negative, so the minimum value is 2 at $x = 3$.

(c) True or false: if $a > b$, then $a^2 > b^2$.

> **False. For example $a = 1, b = -2$.**
> $1 > -2$ but $1^2 = 1 < 4 = (-2)^2$. The statement is true if $a > b > 0$.

(d) True or false: if $a > b$, then $\dfrac{1}{a} < \dfrac{1}{b}$.

> **False. For example $a = 1, b = -1$.**
> $1 > -1$ but $\dfrac{1}{1} = 1 > -1 = \dfrac{1}{-1}$. Both (c) and (d) are true under extra conditions, which are today's properties.

Example 1

Prove that $x^2 + y^2 \geq 2xy$ for all real x and y, using each layout. When does equality hold?

> **Layout 1: $x^2 + y^2 - 2xy = (x-y)^2 \geq 0$. Layout 2: $(x-y)^2 \geq 0 \Rightarrow x^2 - 2xy + y^2 \geq 0 \Rightarrow x^2 + y^2 \geq 2xy$. Equality holds if and only if $x = y$.**
> A square is zero only when what is inside it is zero, so $(x-y)^2 = 0 \Leftrightarrow x = y$. This result is used constantly in Lesson 6 and in HSC questions.

Example 2 · Squares are non-negative

(a) Prove that $a^2 + \dfrac{1}{a^2} \geq 2$ for $a \neq 0$.

> $a^2 + \dfrac{1}{a^2} - 2 = \left(a - \dfrac{1}{a}\right)^2 \geq 0$.
> Expand $\left(a - \frac{1}{a}\right)^2 = a^2 - 2 + \frac{1}{a^2}$. The restriction $a \neq 0$ is needed so that $\frac{1}{a^2}$ exists.

(b) Prove that $(x+y)^2 \geq 4xy$ for all real x, y.

> $(x+y)^2 - 4xy = x^2 - 2xy + y^2 = (x-y)^2 \geq 0$.
> Subtract the right-hand side and recognise a perfect square.

(c) Prove that $\dfrac{a^2+b^2}{2} \geq \left(\dfrac{a+b}{2}\right)^2$.

> $\text{LHS} - \text{RHS} = \dfrac{2a^2+2b^2-a^2-2ab-b^2}{4} = \dfrac{(a-b)^2}{4} \geq 0$.
> In words: the mean of the squares is at least the square of the mean. Use a common denominator of 4 before simplifying.

HSC practice · Squares are non-negative

Let x and y be real numbers with $x^2+y^2 \neq 0$. By considering $(x-y)^2 \geq 0$, show that $\dfrac{(x+y)^2}{x^2+y^2} \leq 2$.

> $(x-y)^2 \geq 0 \Rightarrow x^2+y^2 \geq 2xy \Rightarrow 2(x^2+y^2) \geq x^2+2xy+y^2 = (x+y)^2$. **Since $x^2+y^2 > 0$, divide both sides by it:** $\dfrac{(x+y)^2}{x^2+y^2} \leq 2$.
> State that $x^2+y^2 > 0$ before dividing: it is non-negative and not zero. Dividing by a positive number keeps the inequality sign.

Example 3 · Using the properties

(a) Prove that if $a > b > 0$, then $a^2 > b^2$.

> $a^2 - b^2 = (a-b)(a+b)$, **a product of two positive numbers, so $a^2 > b^2$.**
> $a-b > 0$ because $a > b$, and $a+b > 0$ because both are positive.

(b) Prove that if $a, b > 0$, then $\dfrac{a}{b} + \dfrac{b}{a} \geq 2$.

> $\dfrac{a}{b} + \dfrac{b}{a} - 2 = \dfrac{a^2+b^2-2ab}{ab} = \dfrac{(a-b)^2}{ab} \geq 0$, **since $ab > 0$.**
> Combine over the common denominator ab. A non-negative numerator over a positive denominator is non-negative.

(c) Prove that $a^3+b^3 \geq a^2b + ab^2$ for $a, b > 0$.

> $a^3+b^3-a^2b-ab^2 = a^2(a-b) - b^2(a-b) = (a-b)(a^2-b^2) = (a-b)^2(a+b) \geq 0$.
> Group in pairs and factor out $(a-b)$. Then $(a-b)^2 \geq 0$ and $a+b > 0$, so the product is non-negative.

HSC practice · Which property fails?

Which statement is NOT true for all real numbers $a < b$ (in the domain of each function)?

A. $a^3 < b^3$

B. $e^{-a} > e^{-b}$

C. $\ln a < \ln b$

D. $\dfrac{1}{a} > \dfrac{1}{b}$

> **D.**
> Counterexample: $a = -1, b = 1$ gives $\dfrac{1}{a} = -1 < 1 = \dfrac{1}{b}$. The property $\dfrac{1}{b} > \dfrac{1}{a}$ holds only when a, b are both positive (or both negative). A, B and C hold because x^3 and $\ln x$ are increasing and e^{-x} is decreasing. (Options reworded from the paper.)

Example 4 · Using a given condition

(a) Prove that $ab \leq \dfrac{1}{4}$.

> $1 = (a+b)^2 \geq 4ab$ **(Example 2(b)), so** $ab \leq \dfrac{1}{4}$**.**
> Square the condition $a + b = 1$, then use $(x+y)^2 \geq 4xy$ and divide by 4.

(b) Prove that $a^2 + b^2 \geq \dfrac{1}{2}$.

> $a^2 + b^2 = (a+b)^2 - 2ab = 1 - 2ab \geq 1 - \dfrac{1}{2} = \dfrac{1}{2}$.
> By (a), $2ab \leq \dfrac{1}{2}$, and subtracting a smaller number from 1 gives a larger result.

(c) Hence prove that $\left(1 + \dfrac{1}{a}\right)\left(1 + \dfrac{1}{b}\right) \geq 9$.

> $\left(1 + \dfrac{1}{a}\right)\left(1 + \dfrac{1}{b}\right) = 1 + \dfrac{a+b}{ab} + \dfrac{1}{ab} = 1 + \dfrac{2}{ab} \geq 1 + 2 \times 4 = 9$.
> Expand, then use $a + b = 1$. Since $0 < ab \leq \dfrac{1}{4}$, we have $\dfrac{1}{ab} \geq 4$ (taking reciprocals reverses the inequality).

Exam-style · Prove a result, then use it

(i) Prove that $x^2 + y^2 \geq 2xy$ for all real x and y.

> $x^2 + y^2 - 2xy = (x-y)^2 \geq 0$.
> Layout 1 from Example 1.

(ii) Hence prove that $a^2 + b^2 + c^2 \geq ab + bc + ca$ for all real a, b, c.

> **Add** $a^2 + b^2 \geq 2ab$, $b^2 + c^2 \geq 2bc$ **and** $c^2 + a^2 \geq 2ca$ **to get** $2(a^2 + b^2 + c^2) \geq 2(ab + bc + ca)$**. Halve both sides.**
> Apply (i) three times, once to each pair of variables, then add the three inequalities.

(iii) Hence, or otherwise, show that if $a+b+c=3$, then $ab+bc+ca \le 3$.

> $9=(a+b+c)^2=a^2+b^2+c^2+2(ab+bc+ca)\ge 3(ab+bc+ca)$, **so** $ab+bc+ca\le 3$**.**
> Square the condition, then replace $a^2+b^2+c^2$ by the smaller quantity $ab+bc+ca$ using (ii). This is the syllabus idea of using previously obtained inequalities.

Exit ticket

(a) Prove that $x^2-4x+5>0$ for all real x.

> $x^2-4x+5=(x-2)^2+1\ge 1>0$**.**
> Complete the square. The square is at least 0, so the whole expression is at least 1.

(b) Given $a>b>0$, prove that $\dfrac{a+1}{b+1}<\dfrac{a}{b}$.

> $\dfrac{a}{b}-\dfrac{a+1}{b+1}=\dfrac{a(b+1)-b(a+1)}{b(b+1)}=\dfrac{a-b}{b(b+1)}>0$**.**
> The numerator $a-b>0$ and the denominator $b(b+1)>0$, so the difference is positive.

Spot the flaw · Starting from the result

A student wrote this. Find the flaw, explain why it matters, then write a correct version.
Claim. $a^2+b^2\ge 2ab$ for all real a and b.
1. $a^2+b^2\ge 2ab$
2. $\Rightarrow a^2-2ab+b^2\ge 0$
3. $\Rightarrow (a-b)^2\ge 0$, which is true. Hence $a^2+b^2\ge 2ab$. $\square$

> **The flaw: the student starts from the statement to be proved and ends at something true. A true ending does not make the start true. Correct version:** $(a-b)^2\ge 0\Rightarrow a^2-2ab+b^2\ge 0\Rightarrow a^2+b^2\ge 2ab$ **(Layout 2), or write** $\Leftrightarrow$ **between the lines.**
> From $-1=1$, squaring gives $1=1$, which is true, yet $-1=1$ is false. Here every step happens to be reversible, so the argument can be rescued by writing $\Leftrightarrow$ or by reversing the order. Reversing fails when a step is not reversible, for example squaring both sides, or multiplying by something that may be negative or zero.

Practice worksheet

Squares are non-negative

1. Prove each inequality for all real numbers.

(a) $x^2-6x+10>0$ (b) $a^2+b^2+2\ge 2(a+b)$

(c) $x^2+y^2+1\ge x+y+xy$

> **(a)** $x^2-6x+10=(x-3)^2+1\ge 1>0$**. (b)** $a^2+b^2+2-2a-2b=(a-1)^2+(b-1)^2\ge 0$**. (c)** $2(x^2+y^2+1-x-y-xy)=(x-y)^2+(x-1)^2+(y-1)^2\ge 0$**, so** $x^2+y^2+1-x-y-xy\ge 0$**.**

2. Prove that $a^4 + b^4 \geq a^3b + ab^3$ for all real a and b.

> $a^4 + b^4 - a^3b - ab^3 = a^3(a-b) - b^3(a-b) = (a-b)(a^3 - b^3) = (a-b)^2(a^2 + ab + b^2)$. **Now** $a^2 + ab + b^2 = \left(a + \frac{b}{2}\right)^2 + \frac{3}{4}b^2 \geq 0$**, so the product is non-negative.**

Properties of inequalities

3. Give a counterexample to each statement.

(a) If $a > b$ and $c > d$, then $a - c > b - d$.

(b) If $a > b$ and $c > d$, then $\dfrac{a}{c} > \dfrac{b}{d}$.

> **(a)** $a = 5, b = 1, c = 4, d = 0$**:** $a - c = 1$ **and** $b - d = 1$**, so** $a - c \not> b - d$**. (b)** $a = 2, b = 1, c = 3, d = 1$**:** $a > b$ **and** $c > d$**, but** $\dfrac{a}{c} = \dfrac{2}{3} < 1 = \dfrac{b}{d}$**.**

4. Given $a > b > 0$, prove that $\dfrac{a}{a+1} > \dfrac{b}{b+1}$.

> $\dfrac{a}{a+1} - \dfrac{b}{b+1} = \dfrac{a(b+1) - b(a+1)}{(a+1)(b+1)} = \dfrac{a-b}{(a+1)(b+1)} > 0$**, since** $a - b > 0$ **and both brackets are positive.**

Using a given condition

5. Let $a, b > 0$ with $a + b = 2$. (a) Prove that $ab \leq 1$. (b) Prove that $a^2 + b^2 \geq 2$. (c) Prove that $\dfrac{1}{a} + \dfrac{1}{b} \geq 2$.

> **(a)** $(a+b)^2 - 4ab = (a-b)^2 \geq 0$**, so** $4 \geq 4ab$ **and** $ab \leq 1$**. (b)** $a^2 + b^2 = (a+b)^2 - 2ab = 4 - 2ab \geq 4 - 2 = 2$**. (c)** $\dfrac{1}{a} + \dfrac{1}{b} = \dfrac{a+b}{ab} = \dfrac{2}{ab} \geq 2$**, because** $0 < ab \leq 1$**.**

6. For $a, b, c > 0$, prove that $(a+b+c)\left(\dfrac{1}{a} + \dfrac{1}{b} + \dfrac{1}{c}\right) \geq 9$.

> **Expanding, the product equals** $3 + \left(\dfrac{a}{b} + \dfrac{b}{a}\right) + \left(\dfrac{b}{c} + \dfrac{c}{b}\right) + \left(\dfrac{a}{c} + \dfrac{c}{a}\right)$**. Each bracket is at least 2 (as** $\dfrac{x}{y} + \dfrac{y}{x} \geq 2$ **for positive** x, y**), so the product is at least** $3 + 6 = 9$**.**

Spot the flaw

7. A student "proves" that $x^2 + 1 \geq 2x$ as follows: "$x^2 + 1 \geq 2x \Rightarrow x^2 - 2x + 1 \geq 0 \Rightarrow (x-1)^2 \geq 0$, which is true, so the claim holds." Explain the flaw and write a correct proof.

> **The student starts from the claim and ends at a true statement, which proves nothing by itself. Correct proof:** $x^2 + 1 - 2x = (x-1)^2 \geq 0$**, so** $x^2 + 1 \geq 2x$**.**

Adapted HSC-style

8. (a) Prove that $x^2 + y^2 \geq 2xy$ for all real x, y. [1] (b) Hence show that $(a+b)^2 \leq 2(a^2 + b^2)$ for all real a, b. [1] (c) Hence show that if $a + b + c = 1$, then $a^2 + b^2 + c^2 \geq \dfrac{1}{3}$. [2]

> **(a)** $x^2 + y^2 - 2xy = (x-y)^2 \geq 0$**. (b)** $2(a^2 + b^2) - (a+b)^2 = a^2 - 2ab + b^2 = (a-b)^2 \geq 0$**. (c) Add** $a^2 + b^2 \geq 2ab$**,** $b^2 + c^2 \geq 2bc$**,** $c^2 + a^2 \geq 2ca$ **to get** $a^2 + b^2 + c^2 \geq ab + bc + ca$**. Then** $1 = (a+b+c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ca) \leq 3(a^2 + b^2 + c^2)$**, so** $a^2 + b^2 + c^2 \geq \dfrac{1}{3}$**.**