MCINTYRE MATHS · MATHEMATICS EXTENSION 2

Worked solutions

Lesson 4 · Proof by contrapositive and contradiction

Warm-up · Negations for proofs

(a) The contrapositive of: if n^2 is odd, then n is odd.

> **If n is even, then n^2 is even.**
> The contrapositive of $P \Rightarrow Q$ is $\neg Q \Rightarrow \neg P$: negate both statements and swap them.

(b) The negation of: $\sqrt{2}$ is rational.

> **$\sqrt{2}$ is irrational.**
> For a real number, "not rational" means irrational.

(c) The definition: x is rational means …

> **$x = \dfrac{p}{q}$ with $p, q \in \mathbb{Z}$, $q \neq 0$. We may also take $\dfrac{p}{q}$ in lowest terms (p and q coprime).**
> Cancelling any common factor gives lowest terms. This is the key fact used in proofs of irrationality.

(d) The negation of: there are finitely many primes.

> **There are infinitely many primes.**
> The negation of "finitely many" is "infinitely many".

Example 1

Prove that if n^2 is even, then n is even.

> **Contrapositive: if n is odd, then n^2 is odd.**
> **Let $n = 2k + 1$, $k \in \mathbb{Z}$. Then $n^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$, which is odd.**
> **Hence if n^2 is even, then n is even.**
> A direct proof is awkward: "$n^2 = 2m$" gives no handle on n. The contrapositive starts from "n is odd", which gives a usable form. We will use this result as a lemma later in the lesson.

Example 2 · Proof by contrapositive

For integers n, prove that if $n^2 - 6n + 5$ is even, then n is odd.

> **Contrapositive: if n is even, then $n^2 - 6n + 5$ is odd.**
> **Let $n = 2k$. Then $n^2 - 6n + 5 = 4k^2 - 12k + 5 = 2(2k^2 - 6k + 2) + 1$, which is odd.**
> **Hence if $n^2 - 6n + 5$ is even, then n is odd.**
> The negation of "n is odd" is "n is even", which gives $n = 2k$ to substitute.

Example 3 · Contrapositive with "or"

(a) For integers a and b, prove that if ab is even, then a is even or b is even.

> **Contrapositive: if a and b are both odd, then ab is odd. This was proved in Lesson 3, Example 1: $(2j+1)(2k+1) = 2(2jk+j+k)+1$.**
> Negating "or" gives "and": $\neg(A \vee B) \equiv \neg A \wedge \neg B$. So the negation of "$a$ is even or b is even" is "a is odd and b is odd".

(b) For $x > 0$, prove that if x is irrational, then $\sqrt{x}$ is irrational.

> **Contrapositive: if $\sqrt{x}$ is rational, then x is rational.**
> **Let $\sqrt{x} = \dfrac{p}{q}$ with $p, q \in \mathbb{Z}$, $q \neq 0$. Then $x = \dfrac{p^2}{q^2}$, a ratio of integers with $q^2 \neq 0$, so x is rational.**
> The negation of "irrational" is "rational", and a rational number has a concrete form p/q to work with. That is why the contrapositive is easier.

HSC practice · Proof by contrapositive

Let $n \geq 3$ be an integer. Prove by contrapositive that if $2^n - 1$ is prime, then n is odd.

> **Contrapositive: if $n \geq 3$ is even, then $2^n - 1$ is not prime.**
> **Let $n = 2k$ with $k \geq 2$. Then $2^n - 1 = (2^k)^2 - 1 = (2^k - 1)(2^k + 1)$.**
> **Since $k \geq 2$, $2^k - 1 \geq 3$ and $2^k + 1 \geq 5$, so $2^n - 1$ is a product of two integers greater than 1, and is not prime.**
> **Hence if $2^n - 1$ is prime, then n is odd.**
> The difference of two squares factorises $2^n - 1$ when n is even. Discussion: why $n \geq 3$? For $n = 2$, $2^2 - 1 = 3$ is prime, so the result would fail.

Example 4

Prove that $\sqrt{2}$ is irrational.

> **Suppose, for contradiction, that $\sqrt{2} = \dfrac{p}{q}$ with $p, q \in \mathbb{Z}$, $q \neq 0$, in lowest terms (no common factor).**
> **Then $p^2 = 2q^2$, so p^2 is even and hence p is even (lemma). Let $p = 2m$.**
> **Then $4m^2 = 2q^2$, so $q^2 = 2m^2$, which is even, so q is even.**
> **Both p and q are even, contradicting lowest terms. Hence $\sqrt{2}$ is irrational.**
> Markers look for the explicit assumption ("Suppose, for contradiction, that …") and an explicit statement of the contradiction. Lowest terms is what makes "both even" impossible.

Example 5 · Proof by contradiction

(a) Prove that there are no integers a and b such that $4a + 6b = 13$.

> **Suppose, for contradiction, that such integers exist. Then $4a + 6b = 2(2a + 3b)$ is even, but 13 is odd. This is a contradiction, so no such integers exist.**
> Whenever the claim is "there are no …", assume that there is one and see what goes wrong.

(b) For positive reals x and y, prove that if $xy > 25$, then $x > 5$ or $y > 5$.

> **Suppose, for contradiction, that $xy > 25$ and $x \leq 5$ and $y \leq 5$. Since $x, y > 0$, $xy \leq 5 \times 5 = 25$. This contradicts $xy > 25$. Hence if $xy > 25$, then $x > 5$ or $y > 5$.**
> To contradict $P \Rightarrow Q$ we assume $P \wedge \neg Q$ (Lesson 1). Here $\neg Q$ is "$x \leq 5$ and $y \leq 5$". Positivity is needed to multiply the inequalities.

Example 6 · A logarithm

Prove that $\log_2 5$ is irrational.

> **Suppose, for contradiction, that $\log_2 5 = \dfrac{p}{q}$ with p, q positive integers (valid since $\log_2 5 > 0$).**
> **Then $2^{p/q} = 5$, so $2^p = 5^q$.**
> **The left side is even and the right side is odd. This is a contradiction, so $\log_2 5$ is irrational.**
> Rewrite the logarithm as a power and raise both sides to the power q to remove the fraction. Any positive power of 2 is even and any power of 5 is odd.

HSC practice · Irrational logarithms

Let $n \geq 2$ be an integer. Prove that $\log_n(n+1)$ is irrational.

> **Since $n + 1 > n > 1$, $\log_n(n+1) > 1 > 0$. Suppose, for contradiction, that $\log_n(n+1) = \dfrac{a}{b}$ with a, b positive integers.**
> **Then $n^{a/b} = n + 1$, so $(n+1)^b = n^a$.**
> **One of n and $n+1$ is even and the other is odd. Any positive power of an even number is even and of an odd number is odd, so one side is even and the other is odd. This is a contradiction.**
> **Hence $\log_n(n+1)$ is irrational.**
> You must say why a and b can be taken positive: the logarithm is positive. This step is often missed. Then the parity argument is the same as in Example 6.

HSC practice · Another surd

Prove by contradiction that $\sqrt{23}$ is irrational.

> **Suppose, for contradiction, that $\sqrt{23} = \dfrac{p}{q}$ with $p, q \in \mathbb{Z}$, $q \neq 0$, coprime.**
> **Then $p^2 = 23q^2$, so $23 \mid p^2$. Since 23 is prime, $23 \mid p$. Let $p = 23x$.**
> **Then $23^2x^2 = 23q^2$, so $q^2 = 23x^2$ and $23 \mid q^2$, so $23 \mid q$.**
> **Then 23 divides both p and q, contradicting that they are coprime. Hence $\sqrt{23}$ is irrational.**
> The step "$23 \mid p^2 \Rightarrow 23 \mid p$" uses that 23 is prime. Compare Example 8(a) for 3, where the same fact was proved by contrapositive.

Example 7 · Infinitely many primes (Euclid)

(a) Explain why N is not divisible by any of $p_1, p_2, \ldots, p_n$.

> **Dividing N by any p_i leaves remainder 1.**
> The product $p_1 p_2 \cdots p_n$ is divisible by each p_i, so adding 1 leaves remainder 1.

(b) Explain why this gives a contradiction, and complete the proof.

> **$N > 1$, so N has a prime factor (or is itself prime). That prime is not in the list, contradicting the assumption that the list contains all the primes. Hence there are infinitely many primes.**
> Common error: claiming N itself must be prime. For example, $2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 + 1 = 30031 = 59 \times 509$. What matters is that N has a prime factor outside the list.

Example 8 · Combining methods

(a) Prove by contrapositive that if $3 \mid n^2$, then $3 \mid n$. (Use $n = 3k + 1, 3k + 2$.)

> **Contrapositive: if $3 \nmid n$, then $3 \nmid n^2$. If $3 \nmid n$ then $n = 3k + 1$ or $n = 3k + 2$.**
> **$n = 3k + 1 \Rightarrow n^2 = 3(3k^2 + 2k) + 1$.**
> **$n = 3k + 2 \Rightarrow n^2 = 3(3k^2 + 4k + 1) + 1$.**
> **Neither is divisible by 3. Hence if $3 \mid n^2$, then $3 \mid n$.**
> An integer not divisible by 3 leaves remainder 1 or 2, which gives exactly the two cases.

(b) Hence prove by contradiction that $\sqrt{3}$ is irrational.

> **Suppose $\sqrt{3} = \dfrac{p}{q}$ in lowest terms, $p, q \in \mathbb{Z}$, $q \neq 0$. Then $p^2 = 3q^2$, so $3 \mid p^2$ and, by (a), $3 \mid p$. Let $p = 3m$. Then $9m^2 = 3q^2$, so $q^2 = 3m^2$, so $3 \mid q^2$ and $3 \mid q$. This contradicts lowest terms. Hence $\sqrt{3}$ is irrational.**
> The proof follows the pattern of Example 4, with part (a) acting as the lemma in place of "n^2 even $\Rightarrow n$ even".

(c) Given that $\sqrt{6}$ is irrational, prove that $\sqrt{2} + \sqrt{3}$ is irrational.

> **Suppose, for contradiction, that $\sqrt{2} + \sqrt{3} = r$ with r rational. Squaring, $r^2 = 5 + 2\sqrt{6}$, so $\sqrt{6} = \dfrac{r^2 - 5}{2}$, which is rational. This contradicts the fact that $\sqrt{6}$ is irrational. Hence $\sqrt{2} + \sqrt{3}$ is irrational.**
> Squaring produces the surd $\sqrt{6}$, which is the quantity we are given information about. Rational numbers are closed under addition, multiplication and division by non-zero rationals.

HSC practice · Converse by contrapositive

Let k be an integer. In Lesson 3 you proved: if $3 \mid (k + 1)$, then $3 \mid (k^3 + 1)$.
(i) Write the contrapositive of this statement.
(ii) Write the converse, and prove it is true by proving its contrapositive.

> **(i) If $k^3 + 1$ is not divisible by 3, then $k + 1$ is not divisible by 3.**
> **(ii) Converse: if $3 \mid (k^3 + 1)$, then $3 \mid (k + 1)$. Its contrapositive: if $3 \nmid (k + 1)$, then $3 \nmid (k^3 + 1)$.**
> **If $3 \nmid (k + 1)$, then $k + 1 = 3p + 1$ or $k + 1 = 3p + 2$.**
> **Case 1, $k + 1 = 3p + 1$: $k = 3p$, so $k^3 + 1 = 27p^3 + 1 = 3(9p^3) + 1$.**
> **Case 2, $k + 1 = 3p + 2$: $k = 3p + 1$, so $k^3 + 1 = 27p^3 + 27p^2 + 9p + 2 = 3(9p^3 + 9p^2 + 3p) + 2$.**
> **Neither is divisible by 3, so the converse is true. Hence $3 \mid (k + 1) \Leftrightarrow 3 \mid (k^3 + 1)$.**
> The contrapositive of the converse is the inverse of the original statement. Take care to work with the expression $k^3 + 1$, not $(k + 1)^3$: the quantity that must be shown not divisible by 3 is $k^3 + 1$.

Spot the flaw · Contradiction

Task Find the flaw, explain why it matters, and write a correct version.

> **Flaw 1: "the sum of two irrational numbers is irrational" is false: $\sqrt{2} + (2 - \sqrt{2}) = 2$.**
> **Flaw 2: the assumption in line 1 is never used. Line 2 simply asserts the result, so there is no real contradiction.**
> **Correct version (Example 8(c)): suppose $\sqrt{2} + \sqrt{3} = r \in \mathbb{Q}$. Then $r^2 = 5 + 2\sqrt{6}$, so $\sqrt{6} = \dfrac{r^2 - 5}{2} \in \mathbb{Q}$, contradicting that $\sqrt{6}$ is irrational.**
> A proof by contradiction must use the assumption to reach something impossible. Here the final line only restates the claim, and it relies on a false general statement.

Exit ticket

(a) Prove by contradiction that there is no largest even integer.

> **Suppose, for contradiction, that N is the largest even integer. Then $N + 2$ is even and larger than N. This is a contradiction, so there is no largest even integer.**
> State the assumption, produce a larger even integer, and state the contradiction.

(b) Prove by contrapositive that for integers n, if $4 \mid n^2$, then n is even.

> **Contrapositive: if n is odd, then $4 \nmid n^2$. Let $n = 2k + 1$. Then $n^2 = 4k^2 + 4k + 1 = 4(k^2 + k) + 1$, which leaves remainder 1 on division by 4. Hence $4 \nmid n^2$, and the original statement is true.**
> Assume n is odd and write $n = 2k + 1$. The remainder of 1 shows n^2 is not a multiple of 4.

Practice worksheet

Contrapositive

1. For integers n, prove by contrapositive that if $3n + 2$ is odd, then n is odd.

> **Contrapositive: if n is even, then $3n + 2$ is even. Let $n = 2k$. Then $3n + 2 = 6k + 2 = 2(3k + 1)$, which is even. Hence the original statement is true.**

2. For integers a and b, prove by contrapositive that if $a + b \geq 101$, then $a \geq 51$ or $b \geq 51$.

> **Contrapositive: if $a \leq 50$ and $b \leq 50$, then $a + b \leq 100$. Adding the inequalities gives $a + b \leq 100 < 101$, so $a + b \geq 101$ fails. Hence the original statement is true.**

Contradiction

3. Given that $\sqrt{2}$ is irrational, prove by contradiction that $1 + \sqrt{2}$ is irrational.

> **Suppose $1 + \sqrt{2} = r$ with r rational. Then $\sqrt{2} = r - 1$, which is rational. This contradicts the fact that $\sqrt{2}$ is irrational. Hence $1 + \sqrt{2}$ is irrational.**

4. Prove that there are no integers m and n such that $12m + 9n = 1$.

> **Suppose such integers exist. Then $12m + 9n = 3(4m + 3n)$ is divisible by 3, but 1 is not. This is a contradiction, so no such integers exist.**

5. Prove that there is no smallest positive rational number.

> **Suppose, for contradiction, that $r = \dfrac{p}{q}$ is the smallest positive rational number, with p, q positive integers. Then $\dfrac{r}{2} = \dfrac{p}{2q}$ is also a positive rational number, and $\dfrac{r}{2} < r$. This contradicts the assumption that r is the smallest. Hence there is no smallest positive rational number.**

6. Given that if n^2 is divisible by 5 then n is divisible by 5, prove by contradiction that $\sqrt{5}$ is irrational.

> **Suppose $\sqrt{5} = \dfrac{p}{q}$ in lowest terms, $p, q \in \mathbb{Z}$, $q \neq 0$. Then $p^2 = 5q^2$, so $5 \mid p^2$ and hence $5 \mid p$. Let $p = 5m$. Then $25m^2 = 5q^2$, so $q^2 = 5m^2$, so $5 \mid q^2$ and $5 \mid q$. Both are divisible by 5, contradicting lowest terms. Hence $\sqrt{5}$ is irrational.**

HSC-style

7. Adapted HSC-style. Let n be an integer.

(i) Prove by contrapositive that if n^3 is even, then n is even. **(2 marks)**

(ii) Hence prove by contradiction that $\sqrt[3]{2}$ is irrational. **(2 marks)**

> **(i) Contrapositive: if n is odd, then n^3 is odd. Let $n = 2k + 1$. Then $n^3 = 8k^3 + 12k^2 + 6k + 1 = 2(4k^3 + 6k^2 + 3k) + 1$, which is odd. Hence if n^3 is even, then n is even.**
>
> **(ii) Suppose $\sqrt[3]{2} = \dfrac{p}{q}$ in lowest terms, $p, q \in \mathbb{Z}$, $q \neq 0$. Then $p^3 = 2q^3$, so p^3 is even and by (i) p is even. Let $p = 2m$. Then $8m^3 = 2q^3$, so $q^3 = 4m^3$, which is even, so q is even by (i). Both p and q are even, contradicting lowest terms. Hence $\sqrt[3]{2}$ is irrational.**