MCINTYRE MATHS · MATHEMATICS EXTENSION 2

Worked solutions

Lesson 3 · Number proofs

Warm-up · Algebra we will need

(a) Expand $(2k+1)^2$.

> $4k^2+4k+1 = 2(2k^2+2k)+1$.
> Expanding and then taking a factor of 2 out of the first two terms shows the result has the form "2 × integer + 1", which is how an odd number looks.

(b) Factorise fully $n^3 - n$.

> $n(n-1)(n+1)$.
> Take out the common factor n to get $n(n^2-1)$, then use the difference of two squares. The factors are three consecutive integers.

(c) Show that $9k^2+12k+6$ is a multiple of 3.

> $9k^2+12k+6 = 3(3k^2+4k+2)$.
> Every term is divisible by 3, so 3 factors out and what remains is an integer when k is an integer.

(d) Explain in words why $n(n+1)$ is always even.

> **One of two consecutive integers is even, so the product is even.**
> Consecutive integers alternate between even and odd. A product with an even factor is even.

Example 1 · Odd and even

(a) Prove that the sum of two odd integers is even.

> **Let $a = 2j+1$ and $b = 2k+1$ with $j, k \in \mathbb{Z}$. Then $a+b = 2j+2k+2 = 2(j+k+1)$, which is even since $j+k+1 \in \mathbb{Z}$.**
> Use two different letters, because the two odd integers need not be equal.

(b) Prove that the product of two odd integers is odd.

> **With $a = 2j+1$ and $b = 2k+1$: $ab = 4jk+2j+2k+1 = 2(2jk+j+k)+1$, which is odd since $2jk+j+k \in \mathbb{Z}$.**
> Expand, then group everything except the final 1 as 2 times an integer. Common error: writing $a = 2k+1$ and $b = 2k+1$, which only proves the result for equal numbers.

Example 2 · Parity and divisibility

(a) Prove that if n is odd, then $n^2 + 4n + 7$ is even.

> **Let $n = 2k + 1$, $k \in \mathbb{Z}$. Then $n^2 + 4n + 7 = 4k^2 + 4k + 1 + 8k + 4 + 7 = 4k^2 + 12k + 12 = 2(2k^2 + 6k + 6)$, which is even.**
> Substitute the algebraic form of an odd number, expand, and factor out 2.

(b) Let $3 \mid a$ and $3 \mid b$. Prove that $3 \mid (ma + nb)$ for all integers m and n.

> **Let $a = 3p$ and $b = 3q$ with $p, q \in \mathbb{Z}$. Then $ma + nb = 3mp + 3nq = 3(mp + nq)$, and $mp + nq \in \mathbb{Z}$, so $3 \mid (ma + nb)$.**
> Translate $3 \mid a$ into "$a = 3p$ for some integer p", using a new letter for each divisibility statement.

Example 3

Prove that the square of any integer has the form $3m$ or $3m + 1$, but never $3m + 2$.

> **Let $n \in \mathbb{Z}$.**
> **Case 1, $n = 3k$: $n^2 = 9k^2 = 3(3k^2)$.**
> **Case 2, $n = 3k + 1$: $n^2 = 9k^2 + 6k + 1 = 3(3k^2 + 2k) + 1$.**
> **Case 3, $n = 3k + 2$: $n^2 = 9k^2 + 12k + 4 = 3(3k^2 + 4k + 1) + 1$.**
> **So n^2 is never of the form $3m + 2$.**
> Every integer leaves remainder 0, 1 or 2 on division by 3, so the three cases are exhaustive. We will use the consequence "if $3 \mid n^2$ then $3 \mid n$" in Lesson 4.

Example 4 · Consecutive integers

(a) Prove that $n(n + 1)$ is even for every integer n, by considering n even and n odd.

> **Case 1, $n = 2k$: $n(n + 1) = 2k(2k + 1)$, which is even.**
> **Case 2, $n = 2k + 1$: $n(n + 1) = (2k + 1)(2k + 2) = 2(2k + 1)(k + 1)$, which is even.**
> **In both cases $n(n + 1)$ is even.**
> Every integer is even or odd, so the two cases are exhaustive. In each case a factor of 2 appears.

(b) Hence prove that $n^3 - n$ is divisible by 6 for every integer n.

> **$n^3 - n = (n - 1)n(n + 1)$, a product of three consecutive integers. One of them is a multiple of 3. By part (a), $n(n + 1)$ is even, so the whole product is even. Since 2 and 3 are coprime, $6 \mid n^3 - n$.**
> Divisible by 2 and by 3 gives divisible by $2 \times 3 = 6$ because 2 and 3 are coprime. (For the factor of 3, an alternative is cases $n = 3k, 3k + 1, 3k + 2$.)

HSC practice · Divisible by 8

Let a be an odd integer. By writing $a = 4k + 1$ or $a = 4k + 3$, prove that $a^2 - 1$ is divisible by 8. (Wording adapted from the paper.)

> **Case 1,** $a = 4k + 1$**:** $a^2 - 1 = 16k^2 + 8k = 8(2k^2 + k)$**.**
> **Case 2,** $a = 4k + 3$**:** $a^2 - 1 = 16k^2 + 24k + 8 = 8(2k^2 + 3k + 1)$**.**
> **In both cases** $8 \mid a^2 - 1$**.**
> Every odd integer leaves remainder 1 or 3 on division by 4, so the cases are exhaustive. Alternative: $a^2 - 1 = (a - 1)(a + 1)$ is a product of two consecutive even numbers, and one of these is a multiple of 4.

HSC practice · Odd for every integer

Prove that $(n + 1)^{41} - 79n^{40}$ is odd for every integer n.

> **Case 1,** n **even:** $n + 1$ **is odd, so** $(n + 1)^{41}$ **is odd;** $79n^{40}$ **is even. Odd minus even is odd.**
> **Case 2,** n **odd:** $n + 1$ **is even, so** $(n + 1)^{41}$ **is even;** $79n^{40}$ **is odd** $\times$ **odd, which is odd. Even minus odd is odd.**
> **In all cases the expression is odd.**
> Facts used: odd $\times$ odd is odd (Example 1), any positive power of an odd number is odd, and any positive power of an even number is even. (The original question goes on to deduce that the expression can never equal 2.)

Example 5 · When a claim fails

(a) Disprove the claim with a counterexample.

> **12 is divisible by 4 and by 6, but not by 24.**
> One counterexample is enough to disprove a “for all” claim.

(b) Explain what went wrong, then state and prove a correct version.

> **Correct version: if** $4 \mid n$ **and** $6 \mid n$**, then** $12 \mid n$**.**
> **Proof: let** $n = 4a$ **with** $a \in \mathbb{Z}$**. Since** $6 \mid n$**, we have** $3 \mid 4a$**. As 3 and 4 are coprime,** $3 \mid a$**, so** $a = 3b$ **and** $n = 12b$**. Hence** $12 \mid n$**.**
> 4 and 6 are not coprime: they share the factor 2, so multiplying them ($4 \times 6 = 24$) double-counts it. The correct number is the lowest common multiple, 12.

HSC practice · Divisibility

Let k be an integer. Prove that if $k + 1$ is divisible by 3, then $k^3 + 1$ is divisible by 3.

> $k^3 + 1 = (k + 1)(k^2 - k + 1)$ **(sum of cubes). If** $k + 1 = 3m$ **with** $m \in \mathbb{Z}$**, then** $k^3 + 1 = 3m(k^2 - k + 1)$**, which is divisible by 3.**
> Alternative: $k^3 + 1 = (k + 1)^3 - 3k^2 - 3k = 27m^3 - 3k^2 - 3k = 3(9m^3 - k^2 - k)$. The contrapositive and converse of this result are explored in Lesson 4.

Example 6 · A divisibility test for 7

(a) Show that $10x + y = 10(x - 2y) + 21y$. Hence prove that if $7 \mid (x - 2y)$, then $7 \mid n$.

> $10(x - 2y) + 21y = 10x - 20y + 21y = 10x + y$. **If** $x - 2y = 7m$, **then** $n = 70m + 21y = 7(10m + 3y)$, **so** $7 \mid n$.
> Expand the right-hand side to verify the identity. Then substitute $x - 2y = 7m$ and factor out 7.

(b) Show that $x - 2y = 21x - 2n$. Hence prove the converse.

> $21x - 2(10x + y) = 21x - 20x - 2y = x - 2y$. **If** $n = 7p$, **then** $x - 2y = 21x - 14p = 7(3x - 2p)$, **so** $7 \mid (x - 2y)$.
> The converse says: if $7 \mid n$, then $7 \mid (x - 2y)$. Start from $n = 7p$ and use the identity just shown.

(c) Write the result as an iff statement. Is 3871 divisible by 7?

> $7 \mid n \Leftrightarrow 7 \mid (x - 2y)$. **For 3871:** $387 - 2 = 385$, **then** $38 - 10 = 28 = 7 \times 4$. **So** $7 \mid 3871$ **(indeed** $3871 = 7 \times 553$**).**
> Parts (a) and (b) give both directions, so the statement is an equivalence. Apply the test repeatedly until the number is small enough to check by eye.

Example 7 · Proving two numbers are not equal

Prove that $(5 + 2)^n \neq 5^n + 2^n$ for all integers $n \geq 2$.

> **By the binomial theorem,** $(5 + 2)^n = 5^n + \binom{n}{1}5^{n-1} \cdot 2 + \cdots + 2^n$. **For** $n \geq 2$ **there is at least one middle term and all middle terms are positive, so** $(5 + 2)^n > 5^n + 2^n$. **Hence** $(5 + 2)^n \neq 5^n + 2^n$.
> Proving a strict inequality is an easy way to prove two quantities differ. Discussion: $n = 1$ gives equality, so the condition $n \geq 2$ is needed.

Spot the flaw · Two odd integers

Task Find the flaw, explain why it matters, and write a correct version.

> **Flaw: using the same letter makes the two odd integers equal, so this only proves that the square of an odd integer is odd.**
> **Correct version: let the integers be** $2j + 1$ **and** $2k + 1$ **with** $j, k \in \mathbb{Z}$. **Their product is** $4jk + 2j + 2k + 1 = 2(2jk + j + k) + 1$, **which is odd since** $2jk + j + k \in \mathbb{Z}$.
> Every line of algebra is correct; the proof fails on generality. Two different odd integers need two different letters. Compare Example 1.

Exit ticket

(a) Prove that the sum of any three consecutive integers is divisible by 3.

> **Let the integers be** n, $n + 1$, $n + 2$. **Their sum is** $3n + 3 = 3(n + 1)$, **which is divisible by 3 since** $n + 1 \in \mathbb{Z}$.
> Define the letter, add, factor out 3, and write a conclusion.

(b) Disprove: "The sum of any four consecutive integers is divisible by 4."

> **Take $1+2+3+4=10$, which is not divisible by 4.**
> One counterexample suffices. In general the sum is $n+(n+1)+(n+2)+(n+3)=4n+6$, which always leaves remainder 2 when divided by 4.

Practice worksheet

Direct proofs

1. Prove that the product of an even integer and any integer is even.

> **Let the even integer be $2j$ and the other integer be k, with $j,k\in\mathbb{Z}$. Then the product is $2jk=2(jk)$, which is even since $jk\in\mathbb{Z}$.**

2. Prove that the sum of the squares of two consecutive integers is odd.

> **Let the integers be n and $n+1$. Then $n^2+(n+1)^2=2n^2+2n+1=2(n^2+n)+1$, which is odd since $n^2+n\in\mathbb{Z}$.**

3. Let $3\mid a$ and $3\mid b$. Prove that $3\mid(a^2+5b)$.

> **Let $a=3p$ and $b=3q$ with $p,q\in\mathbb{Z}$. Then $a^2+5b=9p^2+15q=3(3p^2+5q)$, which is divisible by 3 since $3p^2+5q\in\mathbb{Z}$.**

Cases

4. By considering the cases n even and n odd, prove that the square of any integer has the form $4m$ or $4m+1$.

> **Case 1, $n=2k$: $n^2=4k^2=4m$ with $m=k^2$. Case 2, $n=2k+1$: $n^2=4k^2+4k+1=4(k^2+k)+1=4m+1$ with $m=k^2+k$. Every integer is even or odd, so n^2 always has the form $4m$ or $4m+1$.**

Disproving claims and inequalities

5. A student claims: "If a divides bc, then a divides b or a divides c." Disprove the claim with a counterexample, and say for which numbers a the claim would be true.

> **Take $a=6$, $b=2$, $c=3$. Then $bc=6$ is divisible by 6, but 6 divides neither 2 nor 3. (The claim holds when a is prime, but fails for many composite a.)**

6. Prove that $(3+4)^n\neq 3^n+4^n$ for all integers $n\geq 2$.

> **By the binomial theorem, $(3+4)^n=3^n+\binom{n}{1}3^{n-1}\cdot 4+\cdots+4^n$. For $n\geq 2$ the middle terms are all positive, so $(3+4)^n>3^n+4^n$, and hence the two are not equal.**

HSC-style

7. Adapted HSC-style.

(i) Show that $n^3 + 3n^2 + 2n = n(n+1)(n+2)$. **(1 mark)**

(ii) Prove that $n^3 + 3n^2 + 2n$ is divisible by 6 for all integers n. **(2 marks)**

(iii) Disprove: "$n^3 + 3n^2 + 2n$ is divisible by 12 for all integers n." **(1 mark)**

> **(i)** $n^3 + 3n^2 + 2n = n(n^2 + 3n + 2) = n(n+1)(n+2)$**. (ii) This is the product of three consecutive integers, so one is a multiple of 3 and at least one is even. The product is divisible by 2 and 3, which are coprime, so it is divisible by 6. (iii) Take** $n = 1$**:** $1 \times 2 \times 3 = 6$**, which is not divisible by 12.**