MCINTYRE MATHS · MATHEMATICS EXTENSION 2

Worked solutions

Lesson 2 · Quantifiers, converse and contrapositive

Warm-up · Negations from Lesson 1

(a) $x > 3$ and $y \leq 1$

> **$x \leq 3$ or $y > 1$.**
> Negating "and" gives "or", and each part is negated: $\neg(P \wedge Q) \simeq \neg P \vee \neg Q$.

(b) If it rains, then the game is cancelled.

> **It rains and the game is not cancelled.**
> $\neg(P \Rightarrow Q) \simeq P \wedge \neg Q$. An implication only fails when P is true and Q is false.

(c) n is prime or n is even.

> **n is not prime and n is odd.**
> Negating "or" gives "and": both parts must fail. (For an integer, not even means odd.)

(d) True or false: "If $2 + 2 = 5$, then 7 is even."

> **True.**
> The hypothesis is false, so the implication is vacuously true. Indeed $P \Rightarrow Q \simeq \neg P \vee Q$, and $\neg P$ is true.

Lesson 1 recap · Working with symbols

(a) Simplify $\neg(P \vee \neg Q)$.

> $\neg P \wedge Q$
> $\neg(P \vee \neg Q) \simeq \neg P \wedge \neg(\neg Q) \simeq \neg P \wedge Q$, because a double negation cancels.

(b) Show that $\neg(\neg P \vee Q) \simeq P \wedge \neg Q$.

> $\neg(\neg P \vee Q) \simeq \neg(\neg P) \wedge \neg Q \simeq P \wedge \neg Q$
> This is $\neg(P \Rightarrow Q)$, as expected, since $\neg P \vee Q \simeq P \Rightarrow Q$. It confirms the rule for negating "if… then".

(c) "If this is an owl, then it is a bird." Write it using "or", then write its negation using "and".

> **"This is not an owl or it is a bird." Negation: "This is an owl and it is not a bird."**
> Use $P \Rightarrow Q \simeq \neg P \vee Q$, then $\neg(P \Rightarrow Q) \simeq P \wedge \neg Q$.

HSC practice · Negating an implication

Consider: "If n is even and a multiple of 3, then n is a multiple of 6." Which is its negation?

A n is not even or not a multiple of 3, and n is a multiple of 6.

B n is even and a multiple of 3, and n is not a multiple of 6.

C If n is not a multiple of 6, then n is not even or not a multiple of 3.

D If n is not even and not a multiple of 3, then n is not a multiple of 6.

> **B**
> $\neg(P \Rightarrow Q) \simeq P \wedge \neg Q$: keep the hypothesis, deny the conclusion. C is the contrapositive, which is equivalent to the original statement, not its negation. D is the inverse. A negates the wrong part. (Wording adapted from the paper.)

Example 1

Write in English, then simplify: $\forall m \in \mathbb{Z},\ \exists k \in \mathbb{Z}$ such that $(2m+1)^2 = 4k+1$.

> **For all integers m, there exists an integer k such that $(2m+1)^2 = 4k+1$. In English: the square of any odd number is one more than a multiple of 4.**
> Every odd number has the form $2m+1$. Preview of Lesson 3: $(2m+1)^2 = 4m^2+4m+1 = 4m(m+1)+1$, so we can take $k = m(m+1)$.

Example 2

Write in symbols: "Every real number has a square that is not negative." Then write its negation.

> $\forall x \in \mathbb{R},\ x^2 \geq 0$. **Negation: $\exists x \in \mathbb{R}$ such that $x^2 < 0$.**
> This is the same idea as the "all cats are fluffy" example in Lesson 1. The negation is false, because the original statement is true for every real number.

Example 3 · Quantified statements

(a) $\forall n \in \mathbb{N},\ n^2 \geq n$

> **Every natural number is at most its square. True. Negation: $\exists n \in \mathbb{N}$ such that $n^2 < n$.**
> For $n \geq 1$, $n^2 - n = n(n-1) \geq 0$.

(b) $\exists x \in \mathbb{R}$ such that $x^2 = 2$

> **Some real number squares to 2. True. Negation: $\forall x \in \mathbb{R},\ x^2 \neq 2$.**
> $x = \sqrt{2}$ is an example, so the "there exists" statement is true.

(c) $\forall x \in \mathbb{R},\ x^2 > 0$

> **Every real number has a positive square. False. Negation: $\exists x \in \mathbb{R}$ such that $x^2 \leq 0$.**
> $x = 0$ is a counterexample, since $0^2 = 0$ is not greater than 0.

(d) $\exists n \in \mathbb{Z}$ such that $n^2 = 5$

> **Some integer squares to 5. False. Negation: $\forall n \in \mathbb{Z},\ n^2 \neq 5$.**
> The only real solutions of $n^2 = 5$ are $\pm\sqrt{5}$, which are not integers.

HSC practice · Negating "there exists"

Points A, B, P are given. Which is the negation of "There exists a real λ such that $\overrightarrow{AP} = \lambda\overrightarrow{AB}$"?

A There exists a real λ such that $\overrightarrow{AP} \neq \lambda\overrightarrow{AB}$.

B For all real λ, $\overrightarrow{AP} \neq \lambda\overrightarrow{AB}$.

C For all real λ, $\overrightarrow{AP} = \lambda\overrightarrow{AB}$.

D There is no real λ such that $\overrightarrow{AP} \neq \lambda\overrightarrow{AB}$.

B

Negating "there exists" gives "for all", and $=$ becomes $\neq$. Geometrically, P is not on the line AB. D says the same as C ("for all λ, $\overrightarrow{AP} = \lambda\overrightarrow{AB}$"), which is not the negation. A keeps the quantifier unchanged. (Wording adapted from the paper.)

HSC practice · The order of quantifiers

Which statement says that $\dfrac{\ln x}{x^3}$ can be made smaller than any positive number?

A $\exists x > 1$ such that $\forall r > 0$, $\dfrac{\ln x}{x^3} < r$

B $\forall x > 1$, $\exists r > 0$ such that $\dfrac{\ln x}{x^3} < r$

C $\forall r > 0$, $\exists x > 1$ such that $\dfrac{\ln x}{x^3} < r$

D $\exists r > 0$ such that $\forall x > 1$, $\dfrac{\ln x}{x^3} < r$

C

Whatever positive r is chosen first, a suitable x can be found. A is false: it would need $\frac{\ln x}{x^3} \leq 0$ for some $x > 1$, but $\frac{\ln x}{x^3} > 0$ there. B and D say something much weaker (that the values are bounded above). The order of quantifiers matters: "$\forall r\ \exists x$" is not the same as "$\exists x\ \forall r$". (Wording adapted from the paper.)

Example 4

(a) On a die: "If the number is prime, then it is a Fibonacci number." Is it true? Name the sufficient and necessary conditions.

True. Sufficient: the number is prime. Necessary: the number is a Fibonacci number.

The primes on a die are 2, 3, 5 and the Fibonacci numbers on a die are 1, 2, 3, 5, so every prime is Fibonacci. Q does not guarantee P: 1 is Fibonacci but not prime.

(b) Explain why $m > 5$ is sufficient for $m > 3$, and why $m > 3$ is not sufficient for $m > 5$.

Any number greater than 5 is greater than 3, so $m > 5$ is sufficient for $m > 3$. But $m = 4$ satisfies $m > 3$ and not $m > 5$.

So $m > 3$ is not sufficient for $m > 5$ (equivalently, $m > 5$ is not necessary for $m > 3$).

Example 5

(a) Write the converse of "If $x = 3$, then $x^2 = 9$." Is it true?

Converse: if $x^2 = 9$, then $x = 3$. It is false.

$x = -3$ gives $x^2 = 9$ but $x \neq 3$. One counterexample shows the converse is false.

(b) "A triangle is isosceles iff its base angles are equal." Write the two implications.

> **If a triangle is isosceles, then its base angles are equal. If the base angles are equal, then the triangle is isosceles.**
> Both are true, so the equivalence $\Leftrightarrow$ holds. To prove an "iff" statement you must prove both directions.

Example 6 · Converse

(a) If a triangle has two equal sides, then it has two equal angles.

> **Converse: if a triangle has two equal angles, then it has two equal sides. True, so a triangle has two equal sides if and only if it has two equal angles.**
> Both directions are true (the isosceles triangle theorem and its converse).

(b) If Fred murdered Alf, then Alf is dead.

> **Converse: if Alf is dead, then Fred murdered him. False.**
> Alf may have died in some other way, so being dead does not guarantee that Fred murdered him.

(c) $ABCD$ is a square $\Rightarrow$ all angles of $ABCD$ are 90°.

> **Converse: if all angles of $ABCD$ are 90°, then $ABCD$ is a square. False.**
> A rectangle that is not a square is a counterexample.

(d) For $x \in \mathbb{R}$: $x > 2 \Rightarrow x^2 > 4$.

> **Converse: if $x^2 > 4$, then $x > 2$. False.**
> Take $x = -3$: $x^2 = 9 > 4$ but $x \not> 2$.

HSC practice · The converse

Which is the converse of "If a polygon is a square, then it is a rectangle"?

A If a polygon is not a square, then it is not a rectangle.

B If a polygon is not a rectangle, then it is not a square.

C If a polygon is a rectangle, then it is a square.

D A polygon is a square and it is not a rectangle.

> **C**
> The converse swaps the two parts. A is the inverse, B is the contrapositive and D is the negation. (Wording adapted from the paper.)

HSC practice · The converse

P: an animal is a herbivore. Q: the animal eats meat. The statement "If an animal is a herbivore, then it does not eat meat" is $P \Rightarrow \neg Q$. Which is its converse?

A $\neg P \Rightarrow Q$

B $Q \Rightarrow \neg P$

C $P \Rightarrow Q$

D $\neg Q \Rightarrow P$

D

The converse of $P \Rightarrow \neg Q$ swaps the two parts: $\neg Q \Rightarrow P$ ("if an animal does not eat meat, then it is a herbivore"). B is the contrapositive. A is the inverse. C is neither. (Wording adapted from the paper.)

Example 7

P: It is raining. Q: I carry an umbrella. Write $P \Rightarrow Q$, its converse and its contrapositive in words.

Statement: if it is raining, then I carry an umbrella. Converse: if I carry an umbrella, then it is raining. Contrapositive: if I do not carry an umbrella, then it is not raining.

The statement and its contrapositive are equivalent. The converse is not equivalent to the statement: I may carry an umbrella on a sunny day.

Example 8 · Contrapositive

(a) If n^2 is even, then n is even.

If n is odd, then n^2 is odd. True.

An odd number is $2k+1$, and $(2k+1)^2 = 2(2k^2+2k)+1$ is odd. Because the contrapositive is true, so is the original.

(b) If $x > 5$, then $x^2 > 25$.

If $x^2 \leq 25$, then $x \leq 5$. True.

If $x^2 \leq 25$ then $-5 \leq x \leq 5$, so $x \leq 5$.

(c) If a quadrilateral is a rectangle, then its diagonals are equal.

If the diagonals are not equal, then it is not a rectangle. True.

The diagonals of a rectangle are equal, so the contrapositive follows.

(d) If $ab = 0$, then $a = 0$ or $b = 0$.

If $a \neq 0$ and $b \neq 0$, then $ab \neq 0$. True.

Negating "or" gives "and". A product of two non-zero real numbers is non-zero.

HSC practice · The contrapositive

Which is the contrapositive of "For all integers n, if n is a multiple of 6, then n is a multiple of 2"?

A There is an integer n that is a multiple of 6 but not a multiple of 2.

B For all integers n, if n is a multiple of 2, then n is a multiple of 6.

C For all integers n, if n is not a multiple of 2, then n is not a multiple of 6.

D For all integers n, if n is not a multiple of 6, then n is not a multiple of 2.

> **C**
> The contrapositive of $P \Rightarrow Q$ is $\neg Q \Rightarrow \neg P$, and the "for all integers n" stays in front. A is the negation, B is the converse and D is the inverse. (Wording adapted from the paper.)

HSC practice · Which cards?

Each card has a colour on one side and a word on the other. The rule is: "If a card has RED on one side, then it has WIN on the other." The cards show: **RED**, **BLUE**, **WIN**, **TRY AGAIN**. Which cards must be turned over to check the rule? Explain using the contrapositive.

> **RED and TRY AGAIN.**
> RED must be turned over to check $P \Rightarrow Q$ directly: its back must say WIN. TRY AGAIN (not WIN) must be turned over to check the contrapositive $\neg Q \Rightarrow \neg P$ (not WIN $\Rightarrow$ not RED): its back must not be RED. BLUE and WIN cannot break the rule, because WIN with BLUE behind it is allowed. A common error is turning WIN, which tests the converse. (Card labels adapted from the paper.)

Exam practice · Putting it together

(i) Write the statement using quantifier and implication notation.

> $\forall n \in \mathbb{Z},\ (n^2 \text{ is divisible by } 4) \Rightarrow (n \text{ is even})$
> Quantifier first, then the implication.

Exam practice · Putting it together

(ii) Write the converse and prove that the converse is true.

> **Converse: for all integers n, if n is even then n^2 is divisible by 4. Proof: let $n = 2k$ with $k \in \mathbb{Z}$. Then $n^2 = 4k^2$, which is divisible by 4.**
> To prove a "for all" implication, assume the hypothesis for an arbitrary integer and show the conclusion.

Exam practice · Putting it together

(iii) Write the negation of the original statement.

> **There exists an integer n such that n^2 is divisible by 4 and n is odd.**
> The negation of $\forall$ is $\exists$, and $\neg(P \Rightarrow Q) \simeq P \wedge \neg Q$. The original statement is true (it is proved by contrapositive in Lesson 4), so its negation is false.

Spot the flaw · Contrapositive

What is the flaw, and what is a correct version?

> **Line 1 is not the contrapositive. "If $x^2 \leq 4$ then $x \leq 2$" is $\neg P \Rightarrow \neg Q$ (the inverse), which is equivalent to the converse, not to the claim. The claim is false: $x = -3$ gives $x^2 = 9 > 4$ but $x < 2$.**
> The contrapositive of $P \Rightarrow Q$ is $\neg Q \Rightarrow \neg P$, so here it is: if $x \leq 2$, then $x^2 \leq 4$. This is false ($x = -3$), as it must be, since it is equivalent to the claim. Correct version: disprove the claim with the counterexample $x = -3$. (The true statement is the converse: if $x > 2$, then $x^2 > 4$.) Line 2 is correct reasoning about the wrong statement, so always write P and Q out before forming the contrapositive.

Exit ticket

(a) Write $P \Rightarrow Q$, its converse and its contrapositive in words. Which are true?

> **If n is divisible by 6 then n is even: true. Converse: if n is even then n is divisible by 6: false. Contrapositive: if n is odd then n is not divisible by 6: true.**
> The converse fails for $n = 2$. The contrapositive has the same truth value as the original statement.

(b) Write the negation of $P \Rightarrow Q$. Is the negation true or false? Is P sufficient for Q? Is P necessary for Q?

> **Negation: n is divisible by 6 and n is odd. It is false. P is sufficient for Q, but P is not necessary for Q.**
> The negation of a true statement is false. $P \Rightarrow Q$ is true, so P is sufficient for Q. P is not necessary for Q because $n = 2$ is even without being divisible by 6.

Practice worksheet

Converse and equivalence

1. Write the converse of each statement. Is it true? If it is, write an iff statement. If not, give a counterexample.

(a) If a number is divisible by 6, then it is divisible by 3.

(b) If $x = 2$, then $x^2 = 4$.

(c) If a triangle is equilateral, then it is isosceles.

(d) If n is odd, then n^2 is odd.

> **(a) If a number is divisible by 3, then it is divisible by 6. False: 9. (b) If $x^2 = 4$, then $x = 2$. False: $x = -2$. (c) If a triangle is isosceles, then it is equilateral. False: a triangle with sides 5, 5, 6. (d) If n^2 is odd, then n is odd. True, so n is odd iff n^2 is odd.**

Contrapositive

2. Write the contrapositive of each statement.

(a) If n is divisible by 4, then n is even. (b) If $a > 0$ and $b > 0$, then $ab > 0$.

(c) If ab is odd, then a and b are both odd.

> **(a) If n is odd, then n is not divisible by 4. (b) If $ab \leq 0$, then $a \leq 0$ or $b \leq 0$. (c) If a or b is even, then ab is even.**

3. Show that the converse of "If $x > 3$, then $x^2 > 9$" is false, but its contrapositive is true.

> **Converse: if $x^2 > 9$ then $x > 3$. Take $x = -4$: $x^2 = 16 > 9$, but $x \not> 3$, so the converse is false. Contrapositive: if $x^2 \leq 9$ then $x \leq 3$. If $x^2 \leq 9$ then $-3 \leq x \leq 3$, so $x \leq 3$. It is true.**

Quantifiers

4. Write each statement in symbols, decide whether it is true, and write its negation.

(a) Every integer is a rational number. (b) Some real number is not equal to its square.

(c) There is an integer whose square is negative.

> **(a) $\forall n \in \mathbb{Z},\ n \in \mathbb{Q}$. True. Negation: $\exists n \in \mathbb{Z}$ such that $n \notin \mathbb{Q}$. (b) $\exists x \in \mathbb{R}$ such that $x \neq x^2$. True ($x = 2$). Negation: $\forall x \in \mathbb{R},\ x = x^2$. (c) $\exists n \in \mathbb{Z}$ such that $n^2 < 0$. False. Negation: $\forall n \in \mathbb{Z},\ n^2 \geq 0$.**

5. Use a counterexample to show that the statement $\forall x \in \mathbb{R},\ x^2 + x > 0$ is false.

> **Take $x = 0$: $x^2 + x = 0$, which is not greater than 0. (Also $x = -\frac{1}{2}$ gives $-\frac{1}{4}$.)**

6. Which of these is true? Justify your answer. (a) $\forall x \in \mathbb{R},\ \exists y \in \mathbb{R}$ such that $x + y = 0$. (b) $\exists y \in \mathbb{R}$ such that $\forall x \in \mathbb{R},\ x + y = 0$.

> **(a) is true: given any x, choose $y = -x$. (b) is false: no single y works for every x. For example, if y worked then $x = 0$ gives $y = 0$, but $x = 1$ gives $1 + 0 \neq 0$. The order of quantifiers matters.**

Sufficient and necessary

7. Consider: "If a quadrilateral is a square, then it is a rectangle." Is the statement true? Name a sufficient condition and a necessary condition. Is the converse true?

> **True. Being a square is sufficient for being a rectangle, and being a rectangle is necessary for being a square. The converse is false: a 2 by 3 rectangle is not a square, so being a rectangle does not guarantee being a square.**

Adapted HSC-style

8. **Adapted HSC-style.** Consider the statement: "For all integers n, if n is divisible by 6, then n is divisible by 3." (a) Write the converse. (b) Is the converse true? Justify your answer. (c) Write the negation of the original statement.

> **(a) For all integers n, if n is divisible by 3, then n is divisible by 6. (b) False: $n = 3$ is divisible by 3 but not by 6. (c) There exists an integer n such that n is divisible by 6 and n is not divisible by 3.**