MCINTYRE MATHS · MATHEMATICS EXTENSION 2

Worked solutions

Lesson 10 · Calculus and the squeeze theorem

Warm-up

(a) Find the minimum value of $f(x) = e^x - 1 - x$.

> **The minimum value is 0, at $x = 0$. So $e^x \geq 1 + x$ for all x.**
> $f'(x) = e^x - 1 = 0$ at $x = 0$, and $f''(x) = e^x > 0$, so this is a minimum. $f(0) = 1 - 1 - 0 = 0$.

(b) If $h(0) = 0$ and $h'(x) > 0$ for $x > 0$, what can you say about $h(x)$ for $x > 0$?

> $h(x) > 0$ **for $x > 0$.**
> h is increasing for $x > 0$, so $h(x) > h(0) = 0$.

(c) Evaluate $\lim\limits_{x \to \infty} \dfrac{1}{x}$ and $\lim\limits_{x \to 0} x^2$.

> 0 **and** 0.
> $\dfrac{1}{x}$ becomes arbitrarily small as x grows, and $x^2 \to 0^2 = 0$.

(d) Since $-1 \leq \sin x \leq 1$, find bounds for $\dfrac{\sin x}{x}$ when $x > 0$.

> $-\dfrac{1}{x} \leq \dfrac{\sin x}{x} \leq \dfrac{1}{x}$.
> Divide the inequality by $x > 0$; the direction does not change. This is used in Example 5.

Example 1

Prove that $\ln(1 + x) < x$ for all $x > 0$.

> **Let $h(x) = x - \ln(1 + x)$ for $x > 0$. Then $h(0) = 0$ and**
>
> $$h'(x) = 1 - \frac{1}{1 + x} = \frac{x}{1 + x} > 0 \quad \text{for } x > 0.$$
>
> **So h is increasing, and $h(x) > h(0) = 0$ for $x > 0$. Hence $\ln(1 + x) < x$.**
> This is strategy 1. State the domain and give a reason for every sign. Warm-up (a) was strategy 2.

Example 2

(a) Prove that $\sin x < x$ for all $x > 0$.

> **Let $h(x) = x - \sin x$. Then $h(0) = 0$ and $h'(x) = 1 - \cos x \geq 0$, with equality only at isolated points. So h is increasing, and $h(x) > 0$ for $x > 0$.**
> Because $h'(x) = 0$ only at isolated points ($x = 2\pi k$), h is still strictly increasing, so the inequality is strict. Caution: these proofs use $\dfrac{d}{dx}(\sin x) = \cos x$, which is itself proved from $\lim\limits_{x \to 0} \dfrac{\sin x}{x} = 1$. They are fine for proving inequalities, but cannot be used to prove that limit (see Example 6).

(b) Prove that $\cos x \geq 1 - \dfrac{x^2}{2}$ for all real x.

> **Let $h(x) = \cos x - 1 + \dfrac{x^2}{2}$. Then $h(0) = 0$ and $h'(x) = x - \sin x \geq 0$ for $x \geq 0$ by (a). So $h(x) \geq 0$ for $x \geq 0$. Since h is an even function, the result also holds for $x < 0$.**
> Part (a) is used inside part (b): a "previously obtained inequality". For the even function, $h(-x) = \cos(-x) - 1 + \dfrac{(-x)^2}{2} = h(x)$.

HSC practice · From a function inequality to a sequence

(i) Show that $x > \ln x$ for all $x > 0$.
(ii) Hence, or otherwise, show that $e^{n^2+n} > (n!)^2$ for all positive integers n.

> **(i) Let $h(x) = x - \ln x$ for $x > 0$. Then $h'(x) = 1 - \dfrac{1}{x} = 0$ at $x = 1$. Also $h''(x) = \dfrac{1}{x^2} > 0$, so $x = 1$ is the minimum, with $h(1) = 1 > 0$. So $h(x) > 0$ for all $x > 0$.**
> **(ii) By (i), $\ln k < k$ for each $k = 1, 2, \ldots, n$. Then**
>
> $$\ln\left((n!)^2\right) = 2\left(\ln 1 + \ln 2 + \cdots + \ln n\right) < 2(1 + 2 + \cdots + n) = n(n+1) = n^2 + n.$$
>
> **Since $\ln$ is increasing, $(n!)^2 < e^{n^2+n}$.**
> Part (i) is strategy 2 (global minimum). In (ii), apply (i) to each term and use $1 + 2 + \cdots + n = \dfrac{n(n+1)}{2}$. This can also be done by induction. (Wording adapted from the paper.)

Example 3

(a) Use concavity to prove that $\sin x \geq \dfrac{2x}{\pi}$ for $0 \leq x \leq \dfrac{\pi}{2}$.

> **$y = \sin x$ has $y'' = -\sin x \leq 0$ on $\left[0, \dfrac{\pi}{2}\right]$, so it is concave down there. The chord from $(0, 0)$ to $\left(\dfrac{\pi}{2}, 1\right)$ is the line $y = \dfrac{2x}{\pi}$. A concave down curve lies on or above its chord, so $\sin x \geq \dfrac{2x}{\pi}$.**
> Strategy 3. The gradient of the chord is $\dfrac{1 - 0}{\pi/2 - 0} = \dfrac{2}{\pi}$. Sketch both graphs.

(b) By finding the maximum of $f(x) = \dfrac{\ln x}{x}$, prove that $\pi^e < e^\pi$.

> **$f'(x) = \dfrac{1 - \ln x}{x^2} = 0$ at $x = e$. Also $f' > 0$ for $x < e$ and $f' < 0$ for $x > e$, so the maximum is $f(e) = \dfrac{1}{e}$. Since $\pi \neq e$, $\dfrac{\ln \pi}{\pi} < \dfrac{1}{e}$, so $e \ln \pi < \pi$, that is $\ln(\pi^e) < \ln(e^\pi)$. Hence $\pi^e < e^\pi$.**
> Because $x = e$ is the only maximum, $f(\pi)$ is strictly less than $f(e)$. Multiply by $e\pi > 0$, then use $e \ln \pi = \ln(\pi^e)$ and $\pi = \ln(e^\pi)$, and finally the fact that $\ln$ is increasing.

Exam practice · Bounds for the harmonic series

(i) Show that $\dfrac{x}{1+x} < \ln(1+x) < x$ for $x > 0$.
(ii) By putting $x = \dfrac{1}{k}$, show that $\dfrac{1}{k+1} < \ln\left(\dfrac{k+1}{k}\right) < \dfrac{1}{k}$.
(iii) Hence show that $\ln(n+1) < 1 + \dfrac{1}{2} + \dfrac{1}{3} + \cdots + \dfrac{1}{n} < 1 + \ln n$ for all integers $n \geq 2$.

> **(i) The right-hand inequality is Example 1. For the left, let $h(x) = \ln(1+x) - \dfrac{x}{1+x}$. Then $h(0) = 0$ and**
>
> $$h'(x) = \frac{1}{1+x} - \frac{1}{(1+x)^2} = \frac{x}{(1+x)^2} > 0 \quad (x > 0),$$
>
> **so $h(x) > 0$ for $x > 0$.**
> **(ii) $\dfrac{1/k}{1 + 1/k} = \dfrac{1}{k+1}$ and $\ln\left(1 + \dfrac{1}{k}\right) = \ln \dfrac{k+1}{k}$.**
> **(iii) Write $H_n = 1 + \dfrac{1}{2} + \cdots + \dfrac{1}{n}$. Sum $\ln \dfrac{k+1}{k} < \dfrac{1}{k}$ for $k = 1, \ldots, n$: the logs telescope to $\ln(n+1) - \ln 1$, so $\ln(n+1) < H_n$. Sum $\dfrac{1}{k+1} < \ln \dfrac{k+1}{k}$ for $k = 1, \ldots, n-1$: $H_n - 1 < \ln n$, so $H_n < 1 + \ln n$.**
> Telescoping: $\sum_{k=1}^{n} \ln \dfrac{k+1}{k} = \ln \dfrac{2}{1} + \ln \dfrac{3}{2} + \cdots + \ln \dfrac{n+1}{n} = \ln(n+1)$. For the second sum the left side is $\dfrac{1}{2} + \cdots + \dfrac{1}{n} = H_n - 1$. At $n = 1$ the right-hand bound is an equality ($H_1 = 1 = 1 + \ln 1$), so the strict inequality fails; this is why $n \geq 2$. Discussion: $H_n \to \infty$, but very slowly, like $\ln n$.

Example 4

Find $\lim\limits_{x \to 0} x^2 \sin\left(\dfrac{1}{x}\right)$.

> **For $x \neq 0$, $-1 \leq \sin\left(\dfrac{1}{x}\right) \leq 1$. Multiply by $x^2 \geq 0$:**
>
> $$-x^2 \leq x^2 \sin\left(\frac{1}{x}\right) \leq x^2.$$
>
> **Both $-x^2$ and x^2 tend to 0 as $x \to 0$, so by the squeeze theorem the limit is 0.**
> You cannot just substitute $x = 0$, as $\sin\left(\dfrac{1}{x}\right)$ is not defined there and oscillates wildly. Sketch $y = x^2$, $y = -x^2$ and the curve squeezed between them.

Example 5

Use the squeeze theorem to find $\lim_{x\to\infty}\dfrac{\sin x}{x}$.

For $x>0$, $-\dfrac{1}{x}\le\dfrac{\sin x}{x}\le\dfrac{1}{x}$ (warm-up (d)). Both bounds tend to 0 as $x\to\infty$, so $\lim_{x\to\infty}\dfrac{\sin x}{x}=0$.

The curve oscillates between $y=\pm\dfrac{1}{x}$ and is squeezed to the x-axis. Contrast this with $\lim_{x\to 0}\dfrac{\sin x}{x}$, the next example.

Example 6

(i) In the unit circle, $A=(1,0)$, $B=(\cos x,\sin x)$ and C lies on OB extended, on the tangent at A. Compare the areas of $\triangle OAB$, sector OAB and $\triangle OAC$ to prove $\sin x<x<\tan x$ for $0<x<\dfrac{\pi}{2}$.

With radius 1: $\triangle OAB$ has area $\frac{1}{2}\times 1\times 1\times\sin x=\frac{1}{2}\sin x$; sector OAB has area $\frac{1}{2}\times 1^2\times x=\frac{1}{2}x$; $\triangle OAC$ has area $\frac{1}{2}\times OA\times AC=\frac{1}{2}\tan x$. The triangle OAB lies inside the sector, which lies inside $\triangle OAC$, so

$$\tfrac{1}{2}\sin x<\tfrac{1}{2}x<\tfrac{1}{2}\tan x,\quad\text{that is}\quad\sin x<x<\tan x.$$

$AC=\tan x$ because $\triangle OAC$ is right-angled at A with $OA=1$ and angle x at O. The sector area $\frac{1}{2}r^2\theta$ comes from geometry, not calculus, so there is no circularity.

(ii) Hence show that $\cos x<\dfrac{\sin x}{x}<1$ for $0<x<\dfrac{\pi}{2}$.

From $\sin x<x<\tan x$, divide by $\sin x>0$: $1<\dfrac{x}{\sin x}<\dfrac{1}{\cos x}$. Take reciprocals (all positive, so the inequalities reverse): $\cos x<\dfrac{\sin x}{x}<1$.

Dividing $\tan x=\dfrac{\sin x}{\cos x}$ by $\sin x$ gives $\dfrac{1}{\cos x}$. Taking reciprocals of positive numbers reverses the inequality.

(iii) Hence prove that $\lim_{x\to 0}\dfrac{\sin x}{x}=1$.

As $x\to 0^+$, $\cos x\to 1$ and the upper bound is the constant 1, so by the squeeze theorem $\lim_{x\to 0^+}\dfrac{\sin x}{x}=1$. The function $\dfrac{\sin x}{x}$ is even, so the limit from the left is also 1. Hence $\lim_{x\to 0}\dfrac{\sin x}{x}=1$.

This is the limit and the small-angle result used in Extension 1; here is its proof. For the left limit, $\dfrac{\sin(-x)}{-x}=\dfrac{\sin x}{x}$. Discussion: what is a proof allowed to assume?

Spot the flaw · Circular reasoning

Find the flaw and write a correct version.

Flaw: the argument is circular. The result $\frac{d}{dx}(\sin x) = \cos x$ is derived from first principles using $\lim_{h\to 0}\frac{\sin h}{h} = 1$, so line 2 assumes what is being proved.

Correct version: Example 6. Compare areas to get $\sin x < x < \tan x$, then squeeze.

A proof may only assume results that do not depend on the statement being proved. In Extension 1 the derivative of $\sin x$ was taken on trust; Example 6 is the missing foundation.

HSC practice · Squeezing a series to e

It can be shown that for all integers $n \geq 0$:

$$0 \leq 1 - \frac{1}{e}\sum_{r=0}^{n}\frac{1}{r!} \leq \frac{1}{(n+1)!}.$$

Use this to show that $\lim_{n\to\infty}\sum_{r=0}^{n}\frac{1}{r!} = e$.

As $n \to \infty$, $\frac{1}{(n+1)!} \to 0$. The lower bound is 0, so by the squeeze theorem $1 - \frac{1}{e}\sum_{r=0}^{n}\frac{1}{r!} \to 0$.

Therefore $\frac{1}{e}\sum_{r=0}^{n}\frac{1}{r!} \to 1$, that is $\sum_{r=0}^{n}\frac{1}{r!} \to e$.

The squeeze theorem applies to sequences as well as functions. The earlier parts of the original question use integration by parts on $\int_0^1 x^n e^{-x}\,dx$, which we revisit in the Integration unit. (Earlier parts omitted; wording adapted from the paper.)

Which method? · The whole unit

(a) $1 + 3 + 5 + \cdots + (2n-1) = n^2$ for $n \geq 1$.

Method: induction on a series (or the AP sum formula). First line: $P(1)$: $1 = 1^2$.

Assume $\sum_{r=1}^{k}(2r-1) = k^2$, then add the next term: $k^2 + (2k+1) = (k+1)^2$.

(b) $e^x \geq 1 + x$ for all real x.

Method: calculus, global minimum. First line: let $h(x) = e^x - 1 - x$; then $h'(x) = e^x - 1 = 0$ at $x = 0$.

$h''(x) = e^x > 0$, so $h(0) = 0$ is the minimum, hence $h(x) \geq 0$ (warm-up (a)).

(c) $3^n > n^3$ for all integers $n \geq 4$.

Method: induction with an inequality. First line: $P(4)$: $3^4 = 81 > 64 = 4^3$.

Assume $3^k > k^3$ with $k \geq 4$. Then $3^{k+1} > 3k^3 \geq (k+1)^3$, because $\left(1 + \frac{1}{k}\right)^3 \leq \left(\frac{5}{4}\right)^3 < 3$ for $k \geq 4$. (For $n = 3$ the two sides are equal, so the start is $n = 4$.)

(d) $\log_3 7$ is irrational.

> **Method: proof by contradiction. First line: suppose $\log_3 7 = \dfrac{p}{q}$ with p, q positive integers.**
> Then $3^{p/q} = 7$, so $3^p = 7^q$. The left side is divisible by 3 but the right side is not, a contradiction. (A parity argument does not work here, as both sides are odd.)

(e) $\displaystyle\lim_{x\to\infty} \frac{\cos x}{x^2} = 0.$

> **Method: squeeze theorem. First line: for $x > 0$, $-\dfrac{1}{x^2} \le \dfrac{\cos x}{x^2} \le \dfrac{1}{x^2}$.**
> Both bounds tend to 0 as $x \to \infty$, so the limit is 0.

(f) For integers n, if n^3 is even, then n is even.

> **Method: contrapositive. First line: suppose n is odd, so $n = 2k + 1$.**
> Then $n^3 = 8k^3 + 12k^2 + 6k + 1 = 2\left(4k^3 + 6k^2 + 3k\right) + 1$ is odd. So if n is odd then n^3 is odd, which is the contrapositive of the statement.

Exit ticket

Using $\cos x \ge 1 - \dfrac{x^2}{2}$ (Example 2), prove that $\sin x > x - \dfrac{x^3}{6}$ for all $x > 0$.

> **Let $h(x) = \sin x - x + \dfrac{x^3}{6}$. Then $h(0) = 0$ and $h'(x) = \cos x - 1 + \dfrac{x^2}{2} \ge 0$ by Example 2(b), with equality only at isolated points. So h is increasing, and $h(x) > 0$ for $x > 0$, that is $\sin x > x - \dfrac{x^3}{6}$.**
> Same pattern each time: build h, check $h(0) = 0$, and sign h' using the previous inequality. Together with Example 2(a): $x - \dfrac{x^3}{6} < \sin x < x$ for $x > 0$, another squeeze for $\dfrac{\sin x}{x}$.

Practice worksheet

Proving inequalities

1. Prove that $\ln x \le x - 1$ for all $x > 0$.

> **Let $h(x) = x - 1 - \ln x$ for $x > 0$. Then $h'(x) = 1 - \dfrac{1}{x} = 0$ at $x = 1$, and $h''(x) = \dfrac{1}{x^2} > 0$, so $x = 1$ is the minimum. $h(1) = 0$, so $h(x) \ge 0$ for all $x > 0$, that is $\ln x \le x - 1$.**

2. Prove that $e^x > 1 + x + \dfrac{x^2}{2}$ for all $x > 0$.

> **Let $h(x) = e^x - 1 - x - \dfrac{x^2}{2}$. Then $h(0) = 0$ and $h'(x) = e^x - 1 - x$. From the warm-up, $e^x \ge 1 + x$ with equality only at $x = 0$, so $h'(x) > 0$ for $x > 0$. Hence h is increasing and $h(x) > 0$ for $x > 0$.**

3. Using $\sin x > x - \dfrac{x^3}{6}$ for $x > 0$ (the exit ticket result), prove that $\cos x \le 1 - \dfrac{x^2}{2} + \dfrac{x^4}{24}$ for all real x.

> **Let $h(x) = 1 - \dfrac{x^2}{2} + \dfrac{x^4}{24} - \cos x$. Then $h(0) = 0$ and $h'(x) = -x + \dfrac{x^3}{6} + \sin x > 0$ for $x > 0$ by the given inequality. So $h(x) > 0$ for $x > 0$. Since h is even, $h(x) \ge 0$ for all real x.**

4. Show that $f(x) = \dfrac{\ln x}{x}$ is decreasing for $x > e$. Hence show that $5^6 > 6^5$.

> **$f'(x) = \dfrac{1 - \ln x}{x^2} < 0$ for $x > e$ since $\ln x > 1$. So $f(5) > f(6)$: $\dfrac{\ln 5}{5} > \dfrac{\ln 6}{6}$, so $6 \ln 5 > 5 \ln 6$, that is $\ln\left(5^6\right) > \ln\left(6^5\right)$. Hence $5^6 > 6^5$. (Check: $15\,625 > 7\,776$.)**

The squeeze theorem

5. Use the squeeze theorem to find:

(a) $\lim\limits_{x \to 0} x \cos\left(\dfrac{1}{x}\right)$ (b) $\lim\limits_{x \to \infty} \dfrac{2 + \cos x}{x}$

> **(a) $-|x| \le x \cos\left(\dfrac{1}{x}\right) \le |x|$ and $|x| \to 0$, so the limit is 0. (b) For $x > 0$, $1 \le 2 + \cos x \le 3$, so $\dfrac{1}{x} \le \dfrac{2 + \cos x}{x} \le \dfrac{3}{x}$. Both bounds tend to 0, so the limit is 0.**

6. Use $\cos x \ge 1 - \dfrac{x^2}{2}$ to show that $0 \le 1 - \cos x \le \dfrac{x^2}{2}$ for all real x. Hence find $\lim\limits_{x \to 0} \dfrac{1 - \cos x}{x}$.

> **Since $\cos x \le 1$, $1 - \cos x \ge 0$. From $\cos x \ge 1 - \dfrac{x^2}{2}$, $1 - \cos x \le \dfrac{x^2}{2}$. Dividing by $|x| > 0$: $0 \le \left|\dfrac{1 - \cos x}{x}\right| \le \dfrac{|x|}{2}$, so $-\dfrac{|x|}{2} \le \dfrac{1 - \cos x}{x} \le \dfrac{|x|}{2}$. Both bounds tend to 0 as $x \to 0$, so by the squeeze theorem the limit is 0.**

Adapted HSC-style

7. (a) Show that $\ln(1 + x) < x$ for $x > 0$. (b) Hence show that $\left(1 + \dfrac{1}{n}\right)^n < e$ for all positive integers n.

(c) Given that $\ln(1 + x) > \dfrac{x}{1 + x}$ for $x > 0$, show that $\left(1 + \dfrac{1}{n}\right)^{n+1} > e$.

> **(a) Let $h(x) = x - \ln(1 + x)$. Then $h(0) = 0$ and $h'(x) = \dfrac{x}{1 + x} > 0$ for $x > 0$, so $h(x) > 0$. (b) Put $x = \dfrac{1}{n}$: $\ln\left(1 + \dfrac{1}{n}\right) < \dfrac{1}{n}$, so $n \ln\left(1 + \dfrac{1}{n}\right) < 1$, that is $\ln\left(1 + \dfrac{1}{n}\right)^n < 1 = \ln e$. Hence $\left(1 + \dfrac{1}{n}\right)^n < e$. (c) Put $x = \dfrac{1}{n}$: $\ln\left(1 + \dfrac{1}{n}\right) > \dfrac{1/n}{1 + 1/n} = \dfrac{1}{n + 1}$, so $(n + 1) \ln\left(1 + \dfrac{1}{n}\right) > 1$, giving $\left(1 + \dfrac{1}{n}\right)^{n+1} > e$.**