MCINTYRE MATHS · MATHEMATICS EXTENSION 2

Worked solutions

Lesson 1 · Logic and the language of proof

Warm-up · True or false?

(a) All prime numbers are odd.

> **False.**
> 2 is prime and even, so 2 is a counterexample. One counterexample is enough to disprove a "for all" claim.

(b) $n^2 + n + 41$ is prime for every $n \in \mathbb{N}$.

> **False, although it works for $n = 0$ to 39.**
> For $n = 40$: $1600 + 40 + 41 = 1681 = 41^2$. Many examples never prove a statement about infinitely many cases, but one counterexample disproves it.

(c) The sum of two odd numbers is even.

> **True.**
> Checking examples such as $3 + 5 = 8$ is not a proof. We prove this in Lesson 3.

(d) If $x^2 = 9$, then $x = 3$.

> **False.**
> $x = -3$ satisfies $x^2 = 9$ but not $x = 3$, so it is a counterexample.

Example 1 · Which are statements? If so, true or false?

(a) There are kangaroos in Australia.

> **A statement, and it is true.**
> It is a sentence that is either true or false.

(b) Close the door.

> **Not a statement.**
> It is a command, so it is neither true nor false.

(c) $x + 1 = 5$

> **Not a statement until x is known.**
> Its logical value depends on x. It becomes a statement with a quantifier, for example $\exists x \in \mathbb{R}$ such that $x + 1 = 5$, which is true (Lesson 2).

(d) 16 is an odd number.

> **A statement, and it is always false.**
> A statement can be false. What matters is that it has a definite logical value.

(e) History is more interesting than Science.

> **Not a statement.**
> It is an opinion, which has no logical value.

Example 2 · Exhaustion and counterexample

(a) Prove by exhaustion: all odd integers between 16 and 20 are prime.

> **The odd integers between 16 and 20 are 17 and 19. Both are prime, so the statement is proved.**
> There are only two cases, so we can check them all. That is what makes the proof valid.

(b) Disprove: all primes are odd.

> **2 is prime and even, so the statement is false.**
> A single counterexample is enough to disprove a "for all" statement.

HSC practice · A counterexample

Consider the claim: "If n is prime, then $2^n - 1$ is prime." Which value of n is a counterexample?

A $n = 2$ B $n = 3$

C $n = 7$ D $n = 11$

> **D**
> $2^{11} - 1 = 2047 = 23 \times 89$, and 11 is prime. For A, B and C: $2^2 - 1 = 3$, $2^3 - 1 = 7$ and $2^7 - 1 = 127$ are all prime, so they agree with the claim and are not counterexamples. A counterexample must make the "if" part true and the "then" part false.

Example 3 · Negation

1. P: It is raining.

> **It is not raining.**
> The simplest case: $\neg P$ is "not P".

2. Q: All dogs are brown.

> **Not all dogs are brown: there is a dog that is not brown.**
> Do not write "all dogs are not brown". That says no dog is brown, which is too strong.

3. R: The bottle of water is full.

> **The bottle of water is not full.**
> Not "empty". A half-full bottle is neither full nor empty.

4. S: Some students ride to school.

> **No students ride to school.**
> The opposite of "some" is "none".

5. T: $x > 4$

> $x \leq 4$
> Not $x < 4$. The negation must include the boundary case $x = 4$.

6. U: $-3 \leq x \leq 8$

> $x < -3$ **or** $x > 8$
> The "and" hidden in $-3 \leq x \leq 8$ (that is, $x \geq -3$ and $x \leq 8$) becomes "or". This previews the negation rules later in the lesson.

Example 4 · P: 6 is even, Q: 6 is prime

(a) Is $P \wedge Q$ true?

> **No, $P \wedge Q$ is false.**
> P is true and Q is false. Both must be true for "and".

(b) Is $P \vee Q$ true?

> **Yes, $P \vee Q$ is true.**
> At least one must be true, and 6 is even.

Implication

When has Bob lied? Complete the truth table for $P \Rightarrow Q$, $\neg P$ and $\neg P \vee Q$. Then complete: $P \Rightarrow Q$ is false only when …

> **$P \Rightarrow Q$ is false only when P is true and Q is false.**
> Bob only lies when Jane pays and he does not go shopping (row 2). If Jane never pays, Bob cannot have broken his promise, so $P \Rightarrow Q$ is true. This is called being vacuously true. The columns for $P \Rightarrow Q$ and $\neg P \vee Q$ are identical, so $P \Rightarrow Q \simeq \neg P \vee Q$.

P	Q	$P \Rightarrow Q$	$\neg P$	$\neg P \vee Q$
1	1	1	0	1
1	0	0	0	0
0	1	1	1	1
0	0	1	1	1

Example 5 · Testing an implication

P: x is a multiple of 6. Q: x is divisible by 9. Complete a truth table for $P \Rightarrow Q$, giving a value of x for each row. Is the implication always true?

No. $x = 24$ is a counterexample, so $P \Rightarrow Q$ is not always true.

One value of x for each row:

P	Q	Example of x	$P \Rightarrow Q$
1	1	18	1
1	0	24	0
0	1	27	1
0	0	25	1

Rows 3 and 4 are true because P is false (vacuously true). Row 2 is the only row where the implication fails: 24 is a multiple of 6 but not divisible by 9.

Example 6 · Negating compound statements

(a) $x > 2$ and x is even.

$x \leq 2$ or x is odd.

Negating "and" gives "or", and each part is negated: $\neg(P \wedge Q) \simeq \neg P \vee \neg Q$. (For an integer x, "not even" means odd.)

(b) n is odd or n is a multiple of 3.

n is even and n is not a multiple of 3.

Negating "or" gives "and": $\neg(P \vee Q) \simeq \neg P \wedge \neg Q$. Both parts have to fail.

(c) If n is even, then n^2 is even.

n is even and n^2 is odd.

$\neg(P \Rightarrow Q) \simeq P \wedge \neg Q$. Keep the hypothesis and deny the conclusion. Do not write "if n is even then n^2 is odd", which is another implication.

(d) If I live in Tasmania, then I live in Australia.

I live in Tasmania and I do not live in Australia.

Again $P \wedge \neg Q$. (The original is true, so this negation is false for everyone.)

Exit ticket

1. Which is the negation of "If n is prime, then n is odd"?

A n is not prime and n is even.

B n is prime and n is even.

C If n is odd, then n is prime.

D n is not prime or n is odd.

B

$\neg(P \Rightarrow Q) \simeq P \wedge \neg Q$: "$n$ is prime and n is not odd", and not odd means even. A negates both parts. C is the converse. D is equivalent to the original statement ($\neg P \vee Q$).

2. Which is logically equivalent to $P \Rightarrow Q$?

A $Q \Rightarrow P$ B $P \wedge \neg Q$

C $\neg P \vee Q$ D $\neg P \wedge Q$

> **C**
> $P \Rightarrow Q \simeq \neg P \vee Q$, as the truth table shows. B is the negation of $P \Rightarrow Q$. A is the converse. D is false when P and Q are both true, but $P \Rightarrow Q$ is true then.

Practice worksheet

Statements

1. Decide which of these are statements. For each statement, say whether it is true or false.

(a) 7 is a prime number. (b) How old are you?

(c) $2x + 3 = 11$ (d) Every square is a rectangle.

> **(a) A statement, true. (b) Not a statement: it is a question. (c) Not a statement until x is known. (d) A statement, true.**

2. Let P: "n is divisible by 4" and Q: "n is even". For $n = 6$, state whether each is true or false: (a) $P \wedge Q$ (b) $P \vee Q$ (c) $\neg P$ (d) $P \Rightarrow Q$.

> **P is false and Q is true when $n = 6$. (a) False, since P is false. (b) True, since Q is true. (c) True. (d) True, since P is false (an implication with a false hypothesis is true).**

Negation

3. Write the negation of each statement.

(a) All triangles have a right angle. (b) $x \geq 3$

(c) n is prime and n is odd. (d) a is positive or b is negative.

(e) If $x > 0$, then $x^2 > 0$.

> **(a) Not all triangles have a right angle: there is a triangle with no right angle. (b) $x < 3$. (c) n is not prime or n is even. (d) $a \leq 0$ and $b \geq 0$. (e) $x > 0$ and $x^2 \leq 0$.**

4. Complete a truth table (using 1 and 0) to show that $\neg(P \wedge Q) \simeq \neg P \vee \neg Q$.

> **The columns for $\neg(P \wedge Q)$ and $\neg P \vee \neg Q$ are identical, so the statements are logically equivalent.**

P	Q	$P \wedge Q$	$\neg(P \wedge Q)$	$\neg P$	$\neg Q$	$\neg P \vee \neg Q$
1	1	1	0	0	0	0
1	0	0	1	0	1	1
0	1	0	1	1	0	1
0	0	0	1	1	1	1

5. Use $P \Rightarrow Q \simeq \neg P \vee Q$ to show that $\neg(P \Rightarrow Q) \simeq P \wedge \neg Q$.

> $\neg(P \Rightarrow Q) \simeq \neg(\neg P \vee Q) \simeq \neg(\neg P) \wedge \neg Q \simeq P \wedge \neg Q$.

Implication, exhaustion and counterexamples

6. Let P: x is a multiple of 4 and Q: x is a multiple of 8. Find a value of x for each row of the truth table for $P \Rightarrow Q$ (rows P, Q = 1,1; 1,0; 0,1; 0,0). Is $P \Rightarrow Q$ always true?

> **For example: (1,1) $x = 16$, true. (1,0) $x = 12$, false. (0,1) impossible, since every multiple of 8 is a multiple of 4. (0,0) $x = 5$, true. So $P \Rightarrow Q$ is not always true: $x = 12$ is a counterexample.**

7. (a) Prove by exhaustion: all prime numbers between 20 and 30 are odd. (b) Use a counterexample to show that the statement "For all real x, $x^2 > x$" is false.

> **(a) The primes between 20 and 30 are 23 and 29. Both are odd, so the statement is proved. (b) Take $x = 1$: $x^2 = 1$, which is not greater than $x = 1$. (Also $x = 0$ or $x = \frac{1}{2}$ work.) One counterexample disproves a "for all" statement.**

Adapted HSC-style

8. **Adapted HSC-style.** Consider: "If n is a multiple of 4 and a multiple of 6, then n is a multiple of 24." (a) Write the negation. (b) Is the original statement true? Justify your answer.

> **(a) n is a multiple of 4 and a multiple of 6, and n is not a multiple of 24. (b) False. Take $n = 12$: it is a multiple of 4 and of 6, but not of 24. So the negation is true.**