MCINTYRE MATHS · MATHEMATICS EXTENSION 2

Worked solutions

Lesson 9 · Lines and circles in the Argand diagram

Retrieval and coordinate geometry

(a) Write the equation of the circle with centre $(1, -2)$ and radius 3.

> $(x-1)^2 + (y+2)^2 = 9$
> A circle with centre (a, b) and radius r has equation $(x-a)^2 + (y-b)^2 = r^2$.

(b) Find the perpendicular bisector of the interval from $(0, 0)$ to $(4, 2)$.

> $y = -2x + 5$
> The midpoint is $(2, 1)$ and the interval has gradient $\frac{1}{2}$, so the bisector has gradient -2: $y - 1 = -2(x-2)$.

(c) Find $|3 - 4i|$.

> 5
> $\sqrt{3^2 + (-4)^2} = \sqrt{25} = 5$.

(d) Write $z + 2 - i$ in the form $z - (\quad)$.

> $z - (-2 + i)$
> $z + 2 - i = z - (-2 + i)$. Parts (a), (b) and (d) are the three skills this lesson needs.

Example 1 · Sketching circles

(a) $|z| = 3$

> $x^2 + y^2 = 9$**: centre O, radius 3.**
> $|z - 0| = 3$: the distance from O is always 3.

(b) $|z - (1 - 2i)| = 2$

> $(x-1)^2 + (y+2)^2 = 4$**: centre $(1, -2)$, radius 2.**
> The centre is 1 unit from the imaginary axis and the radius is 2, so the circle cuts the imaginary axis.

Example 2 · Regions bounded by circles

(a) $|z + 3 - i| \leq 2$

> **The closed disc with centre $(-3, 1)$ and radius 2, solid boundary, shaded inside.**
> Rewrite as $|z - (-3 + i)| \leq 2$. The inequality is "$\leq$", so the circle itself is included.

(b) $|z-2i|>1$

> **Everything outside the circle with centre $(0,2)$ and radius 1, with a dashed boundary.**
> $|z-2i|>1$ means the distance from $2i$ is more than 1. The inequality is strict, so the circle is not included.

Example 3

Sketch $|z-(1+i)|=|z-(2-i)|$ and find its Cartesian equation using two methods.

> $2x-4y-3=0$
> **Method A.** Midpoint $\left(\frac{3}{2},0\right)$. Gradient of Z_1Z_2 is $\frac{-1-1}{2-1}=-2$, so the bisector has gradient $\frac{1}{2}$: $y=\frac{1}{2}\left(x-\frac{3}{2}\right)$, giving $2x-4y-3=0$.
> **Method B.** $(x-1)^2+(y-1)^2=(x-2)^2+(y+1)^2$ gives $-2x-2y+2=-4x+2y+5$, so $2x-4y-3=0$.

Example 4 · A circle and a line

(i) On the same diagram sketch $|z-(3+2i)|=2$ and $|z+3|=|z-5|$. (ii) Find the complex number(s) satisfying both equations.

> **(i) A circle with centre $(3,2)$ and radius 2, and the vertical line $x=1$ (the perpendicular bisector of -3 and 5). (ii) The line is tangent to the circle: $z=1+2i$.**
> The leftmost point of the circle is $x=3-2=1$, so the line $x=1$ touches the circle at exactly one point, $(1,2)$.

Example 5 · Touching circles

For what values of $k>0$ does the circle $|z-2i|=k$ meet $|z-(3+2i)|=2$ at exactly one point?

> **$k=1$ or $k=5$.**
> The centres $(0,2)$ and $(3,2)$ are 3 apart. External tangency: $k+2=3$, so $k=1$. Internal tangency (the smaller circle inside the larger): $k-2=3$, so $k=5$.

Example 6

Find the locus of w if $w=\dfrac{z-1}{z}$ and $|z|=2$.

> **The circle $|w-1|=\dfrac{1}{2}$: centre 1, radius $\dfrac{1}{2}$.**
> Make z the subject: $wz=z-1$, so $z(w-1)=-1$ and $z=\dfrac{-1}{w-1}$. Then $|z|=2$ gives $\dfrac{1}{|w-1|}=2$, so $|w-1|=\frac{1}{2}$. In Cartesian form, $(u-1)^2+v^2=\frac{1}{4}$ for $w=u+iv$. **Strategy:** if w is given in terms of z, make z the subject, then apply the condition on z.

Example 7

Find the locus of z if $w = \dfrac{z-2}{z}$ is purely imaginary.

The circle $(x-1)^2 + y^2 = 1$, with the points $z = 0$ and $z = 2$ excluded.

$w = \dfrac{(x-2+iy)(x-iy)}{x^2+y^2} = \dfrac{x^2+y^2-2x+2iy}{x^2+y^2}$. Purely imaginary means $\mathrm{Re}(w) = 0$ (and $w \neq 0$), so $x^2+y^2-2x = 0$, that is $(x-1)^2+y^2 = 1$. Exclude $z = 0$ (where w is undefined) and $z = 2$ (where $w = 0$, which is real). Mark both with open circles.

Example 8

Describe the locus given by $|z+2| = 2|z-2+i|$.

A circle with centre $\left(\dfrac{10}{3}, -\dfrac{4}{3}\right)$ and radius $\dfrac{2\sqrt{17}}{3}$.

$(x+2)^2 + y^2 = 4\left[(x-2)^2 + (y+1)^2\right]$ gives $3x^2+3y^2-20x+8y+16 = 0$, so $\left(x-\frac{10}{3}\right)^2 + \left(y+\frac{4}{3}\right)^2 = \frac{68}{9}$. When the ratio of distances is not 1, the locus is a circle, not a line.

Spot the error · Circles and bisectors

(a) $|z+3-i| = 2$ is a circle with centre $(3, -1)$.

The centre is $(-3, 1)$.

Write $|z+3-i| = |z-(-3+i)|$, so the centre is $-3+i$, that is $(-3, 1)$. The sign flips when you read off the centre.

(b) $|z-2i| < 1$ is drawn with a solid boundary.

The boundary should be dashed.

The inequality is strict, so the points on the circle are not in the region.

(c) $|2z-4i| = 6$ is a circle with radius 6.

The radius is 3 and the centre is $2i$.

$|2z-4i| = 2|z-2i| = 6$, so $|z-2i| = 3$. Always factor out the coefficient of z first.

(d) $|z-1| = |z-i|$ is the circle through 1 and i.

It is a line, not a circle: the line $y = x$.

Equal distances from 1 and i give the perpendicular bisector of the interval joining them, which is $y = x$.

Exit ticket

1. Sketch $|z-1-i| = |z+1+i|$ and give its equation.

The line $y = -x$.

This is the perpendicular bisector of $(1, 1)$ and $(-1, -1)$. The midpoint is O and the interval has gradient 1, so the bisector passes through O with gradient -1.

2. Write the Cartesian equation of $|2z - 4i| = 6$ and sketch it.

> $x^2 + (y-2)^2 = 9$.
> Divide by 2: $|z - 2i| = 3$, a circle with centre $(0, 2)$ and radius 3.

Practice worksheet

Circles and regions

1. Sketch each locus or region and write its Cartesian equation.

(a) $|z - 2 + i| = 3$ (b) $|z + 1| \leq 1$

(c) $|z - 2i| > 2$

> **(a) Circle, centre $(2, -1)$, radius 3: $(x-2)^2 + (y+1)^2 = 9$. (b) Closed disc, centre $(-1, 0)$, radius 1: $(x+1)^2 + y^2 \leq 1$. (c) Outside the dashed circle with centre $(0, 2)$, radius 2: $x^2 + (y-2)^2 > 4$.**

Perpendicular bisectors

2. Find the Cartesian equation of $|z - 3| = |z + i|$.

> **$(x-3)^2 + y^2 = x^2 + (y+1)^2$ gives $-6x + 9 = 2y + 1$, so $3x + y - 4 = 0$.**

3. The circle $|z - 1 - i| = k$ touches the locus $|z - 1| = |z - 3|$. Find k and the point of contact.

> **The locus is the line $x = 2$. The centre $(1, 1)$ is 1 unit from it, so $k = 1$ and the point of contact is $2 + i$.**

Change of variable

4. If $|z - 1| = 1$ and $w = \dfrac{1}{z}$, find the locus of w.

> **$z = \frac{1}{w}$, so $\left|\frac{1}{w} - 1\right| = 1$, giving $|1 - w| = |w|$. The locus is the perpendicular bisector of 0 and 1: the line $\mathrm{Re}(w) = \frac{1}{2}$.**

5. Find the locus of z if $\dfrac{z-1}{z+1}$ is purely imaginary.

> $\dfrac{(x-1+iy)(x+1-iy)}{(x+1)^2+y^2}$ **has real part** $\dfrac{x^2+y^2-1}{(x+1)^2+y^2}$**. Setting this to 0 gives the circle $x^2 + y^2 = 1$ with $z = \pm 1$ excluded.**

6. If $w = 2z + 1$ and $|z| = 3$, find the locus of w.

> **$z = \frac{w-1}{2}$, so $|w - 1| = 6$: a circle with centre 1 and radius 6.**

HSC-style

7. (a) On one Argand diagram sketch $|z - (1+i)| = 2$ and $|z - 2| = |z + 2i|$. (b) Find the complex numbers that satisfy both equations.

> **(a) A circle with centre $(1, 1)$ and radius 2, and the line $y = -x$ (the perpendicular bisector of 2 and $-2i$, through the midpoint $(1, -1)$ with gradient -1). (b) Substituting $y = -x$: $(x-1)^2 + (x+1)^2 = 4$, so $x^2 = 1$ and $z = 1 - i$ or $z = -1 + i$.**