MCINTYRE MATHS · MATHEMATICS EXTENSION 2

Worked solutions

Lesson 8 · Rotation, dilation and geometry

Example 1

Let $z = 2 + i$. Plot z, iz, $i^2z = -z$ and $i^3z = -iz$. Describe what multiplying by i does.

> $iz = -1 + 2i$, $-z = -2 - i$, $-iz = 1 - 2i$. **Multiplying by i rotates a point 90° anticlockwise about O.**
> Each multiplication by $i = \operatorname{cis}\frac{\pi}{2}$ keeps the modulus and adds $\frac{\pi}{2}$ to the argument. The four points form a square centred at O.

Warm-up · Retrieval from Lessons 5 to 7

(a) Find iz for $z = 3 - 2i$.

> $iz = 2 + 3i$
> $i(3 - 2i) = 3i - 2i^2 = 2 + 3i$, using $i^2 = -1$.

(b) Write $\cos\dfrac{\pi}{3} + i\sin\dfrac{\pi}{3}$ as $x + iy$.

> $\dfrac{1}{2} + \dfrac{\sqrt{3}}{2}i$
> Use the exact values $\cos\frac{\pi}{3} = \frac{1}{2}$ and $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$.

(c) Z is $1 + i$ and W is $3 + 4i$. Which complex number does $\overrightarrow{WZ}$ represent?

> $z - w = -2 - 3i$
> Head minus tail: the vector from W to Z represents $z - w$.

(d) Find the length WZ in part (c).

> $WZ = \sqrt{13}$
> $WZ = |z - w| = |-2 - 3i| = \sqrt{4 + 9} = \sqrt{13}$.

Example 2

Let $z = 2 + 2i$. Find the complex number obtained by rotating z anticlockwise by $\dfrac{\pi}{3}$ about O and doubling its distance from O.

> $(2 - 2\sqrt{3}) + (2 + 2\sqrt{3})i$
> Multiply by $2\operatorname{cis}\frac{\pi}{3} = 1 + \sqrt{3}\,i$: $(2 + 2i)(1 + \sqrt{3}i) = 2 + 2\sqrt{3}i + 2i - 2\sqrt{3}$. Check: $|z| = 2\sqrt{2}$ becomes $4\sqrt{2}$, and $\arg z = \frac{\pi}{4}$ becomes $\frac{\pi}{4} + \frac{\pi}{3} = \frac{7\pi}{12}$.

Example 3 · A square

(a) Find C and B.

> **$C = -1 + 2i$ and $B = 1 + 3i$.**
> Rotating $\overrightarrow{OA}$ by 90° anticlockwise gives $\overrightarrow{OC}$: $c = i(2 + i) = -1 + 2i$. Then $b = a + c = (2 + i) + (-1 + 2i) = 1 + 3i$ since $OABC$ is a parallelogram.

(b) The square is rotated 45° anticlockwise about O. Find the new position of B.

> $-\sqrt{2} + 2\sqrt{2}\,i$
> $b' = b\,\mathrm{cis}\,\frac{\pi}{4} = (1 + 3i) \cdot \frac{1+i}{\sqrt{2}} = \frac{-2+4i}{\sqrt{2}} = -\sqrt{2} + 2\sqrt{2}\,i.$

Example 4

In triangle ABC, $AB = BC$ and $\angle ABC = 90^\circ$. B is $5 + 3i$ and C is $9 + 6i$, and A lies above the line BC. Find the complex number represented by A.

> $A = 2 + 7i$
> $\overrightarrow{BC} = c - b = 4 + 3i$. Since $BA = BC$ and $\angle ABC = 90^\circ$ with A above BC, $\overrightarrow{BA}$ is $\overrightarrow{BC}$ rotated 90° anticlockwise: $i(4 + 3i) = -3 + 4i$. So $a = b + (-3 + 4i) = 2 + 7i$. (If A were below BC, multiply by $-i$ to get $a = 8 - i$.)

Example 5 · Equilateral triangle

(i) Explain why $\triangle OAB$ is equilateral. (ii) Show that $z_1^2 - z_1 z_2 + z_2^2 = 0$.

> **(i) $|z_2| = |z_1|$, so $OA = OB$, and $\angle AOB = \frac{\pi}{3}$. An isosceles triangle with apex angle 60° is equilateral. (ii) Let $\omega = \mathrm{cis}\,\frac{\pi}{3}$. Then $\omega^3 = \mathrm{cis}\,\pi = -1$, so $(\omega + 1)(\omega^2 - \omega + 1) = 0$. Since $\omega \neq -1$, $\omega^2 - \omega + 1 = 0$, and $z_1^2 - z_1 z_2 + z_2^2 = z_1^2(1 - \omega + \omega^2) = 0$.**
> Alternatively, $\omega^2 = \mathrm{cis}\,\frac{2\pi}{3} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$ and $\omega - 1 = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$ are equal, so $\omega^2 = \omega - 1$. (The original paper writes z_2 in exponential form.)

Example 6 · Perpendicular vectors

(i) Prove that $\left|\dfrac{z - w}{z + w}\right| = 1$. (ii) Interpret this result geometrically.

> **(i) $\frac{z}{w} = -ri$ for some $r > 0$. Dividing top and bottom by w: $\dfrac{z - w}{z + w} = \dfrac{z/w - 1}{z/w + 1} = \dfrac{-1 - ri}{1 - ri}$. Both numerator and denominator have modulus $\sqrt{1 + r^2}$, so the modulus is 1. (ii) $OZ \perp OW$, so the parallelogram on OZ and OW is a rectangle and its diagonals $|z + w|$ and $|z - w|$ are equal.**
> $\arg\frac{z}{w} = -\frac{\pi}{2}$ means $\frac{z}{w}$ is a negative imaginary number. Rotating w by -90° gives a vector in the direction of z, so $OZ \perp OW$ (compare the exit ticket in Lesson 7).

Example 7 · Proving a quadrilateral is a square

(a) If $z_1 - z_2 = z_4 - z_3$, show that $ABCD$ is a parallelogram.

> **$z_1 - z_2$ represents $\overrightarrow{BA}$ and $z_4 - z_3$ represents $\overrightarrow{CD}$. Equal vectors mean $BA \parallel CD$ and $BA = CD$, so $ABCD$ is a parallelogram.**
> Name the vector each complex number represents before interpreting it. One pair of sides equal and parallel is enough for a parallelogram.

(b) If also $z_3 - z_2 = i(z_1 - z_2)$, show that $ABCD$ is a square.

> **$z_3 - z_2 = \overrightarrow{BC}$ is $\overrightarrow{BA}$ rotated through $90°$, so $BC = BA$ and $\angle ABC = 90°$. A parallelogram with a right angle and two equal adjacent sides is a square.**
> Multiplying by i keeps the length and rotates by $90°$.

Exit ticket

1. O, 2 and z form an equilateral triangle with z in the first quadrant. Find z.

> $z = 1 + \sqrt{3}\,i$
> $z = 2\,\mathrm{cis}\,\frac{\pi}{3}$: rotate the vector 2 about O through $60°$ anticlockwise (the third vertex is at the same distance, 2, from O).

2. $ABCD$ is a square labelled anticlockwise, with A at $1 + i$ and B at $4 + 2i$. Find C and D.

> **$C = 3 + 5i$ and $D = 4i$.**
> $\overrightarrow{AB} = 3 + i$. $\overrightarrow{BC} = i(3 + i) = -1 + 3i$, so $C = (4 + 2i) + (-1 + 3i) = 3 + 5i$. $D = A + \overrightarrow{BC} = (1 + i) + (-1 + 3i) = 4i$.

Practice worksheet

Rotation and dilation

1. Find the image of $z = 3 - i$ when it is (a) rotated $90°$ anticlockwise about O, (b) rotated $90°$ clockwise about O, (c) rotated $60°$ anticlockwise about O and tripled in distance from O.

> **(a)** $i(3 - i) = 1 + 3i$. **(b)** $-i(3 - i) = -1 - 3i$. **(c)** $3\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)(3 - i) = \frac{3}{2}(1 + \sqrt{3}i)(3 - i) = \frac{3}{2}\big((3 + \sqrt{3}) + (3\sqrt{3} - 1)i\big)$.

2. The point P represents $4 + i$. Find the point Q obtained by rotating P through $\dfrac{\pi}{2}$ anticlockwise about the point A representing $1 + 2i$.

> $q - a = i(p - a) = i(3 - i) = 1 + 3i$, **so** $q = 2 + 5i$.

Vertices of shapes

3. $OABC$ is a square, labelled anticlockwise, with B at $4 + 6i$. Find A and C.

> **The centre is $2 + 3i$ and $\overrightarrow{OA}$ is $\overrightarrow{OB}$ rotated $-45°$ and scaled by $\frac{1}{\sqrt{2}}$: $a = (4 + 6i) \cdot \frac{1-i}{2} = 5 + i$. Then $c = ia = -1 + 5i$. Check: $a + c = 4 + 6i$.**

4. A and B represent $1+i$ and $5+3i$. Find the third vertex C of the equilateral triangle ABC lying above AB (anticlockwise order A, B, C).

> $\overrightarrow{AB} = 4+2i$. $\overrightarrow{AC} = \overrightarrow{AB}\,\mathrm{cis}\,\frac{\pi}{3} = (4+2i)\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right) = (2-\sqrt{3})+(1+2\sqrt{3})i$. So $C = (3-\sqrt{3})+(2+2\sqrt{3})i$.

Proofs

5. Show that if $z_2 = iz_1$ then z_1, z_2 and the origin form an isosceles right-angled triangle.

> $|z_2| = |i||z_1| = |z_1|$, so $OZ_1 = OZ_2$. $\arg z_2 = \arg z_1 + \frac{\pi}{2}$, so $\angle Z_1OZ_2 = 90^\circ$.

6. A, B, C represent z_1, z_2, z_3. If $z_3 - z_1 = i(z_2 - z_1)$, show that $\triangle ABC$ is right-angled and isosceles, and state the right angle.

> $\overrightarrow{AC} = i\,\overrightarrow{AB}$, so $AC = AB$ and $\angle BAC = 90^\circ$. The right angle is at A.

HSC-style

7. Adapted HSC-style. The complex numbers 0, z and w are the vertices of an equilateral triangle, labelled anticlockwise. (a) Show that $w = z\,\mathrm{cis}\,\frac{\pi}{3}$. (b) Show that $z^2 + w^2 = zw$. (c) Hence show that $\frac{z}{w} + \frac{w}{z} = 1$.

> (a) $OW = OZ$ and $\angle ZOW = \frac{\pi}{3}$ anticlockwise, so $w = z\,\mathrm{cis}\,\frac{\pi}{3}$. (b) With $\omega = \mathrm{cis}\,\frac{\pi}{3}$, $\omega^2 - \omega + 1 = 0$ (since $\omega^3 = -1$, $\omega \neq -1$), so $z^2 + w^2 - zw = z^2(1+\omega^2-\omega) = 0$. (c) Divide $z^2 + w^2 = zw$ by $zw \neq 0$.