MCINTYRE MATHS · MATHEMATICS EXTENSION 2

# Worked solutions

Lesson 7 · Complex numbers as vectors

## Warm-up · Retrieval from Lessons 4 to 6

**(a)** Find $|z|$ if $z = (3+4i)(1-i)$.

> $|z| = 5\sqrt{2}$
> The modulus of a product is the product of the moduli: $|3+4i||1-i| = 5 \cdot \sqrt{2}$.

**(b)** Find $\operatorname{Arg}(z_1 z_2)$ if $\arg z_1 = \dfrac{3\pi}{4}$ and $\arg z_2 = \dfrac{\pi}{2}$.

> $\operatorname{Arg}(z_1 z_2) = -\dfrac{3\pi}{4}$
> Arguments add: $\frac{3\pi}{4} + \frac{\pi}{2} = \frac{5\pi}{4}$. This is outside $(-\pi, \pi]$, so subtract $2\pi$ to get the principal argument $-\frac{3\pi}{4}$.

**(c)** For $\underset{\sim}{a} = 3\underset{\sim}{i} + 2\underset{\sim}{j}$ and $\underset{\sim}{b} = \underset{\sim}{i} - 2\underset{\sim}{j}$, find $\underset{\sim}{a} + \underset{\sim}{b}$ and $\underset{\sim}{a} - \underset{\sim}{b}$.

> $\underset{\sim}{a} + \underset{\sim}{b} = 4\underset{\sim}{i}$ **and** $\underset{\sim}{a} - \underset{\sim}{b} = 2\underset{\sim}{i} + 4\underset{\sim}{j}$**.**
> Add or subtract components. These are the same numbers as $u + v = 4$ and $u - v = 2 + 4i$ in Example 3, which you met in the Extension 1 vectors work.

**(d)** Plot $z = -1 + 2i$ and $iz$. Through what angle has $z$ turned about $O$?

> $iz = -2 - i$**, and $z$ has turned through $90°$ anticlockwise.**
> $iz = i(-1+2i) = -i - 2$. Multiplying by $i = \operatorname{cis}\frac{\pi}{2}$ adds $\frac{\pi}{2}$ to the argument.

## Example 1

Plot $z = 3 + 4i$ and $\overline{z}$ as vectors from $O$. Draw the vector from $Z$ to $\overline{Z}$. Which complex number does it represent?

> $\overline{z} - z = (3-4i) - (3+4i) = -8i$**: a vector pointing straight down, of length 8.**
> The vector from $Z$ to $\overline{Z}$ is "head minus tail", $\overline{z} - z$. The same vector $-8i$ can be drawn starting anywhere: only length and direction matter.

## Example 2 · Adding on the Argand diagram

**(a)** Show $(1+2i)+(3+i)$ on the Argand diagram.

> $(1+2i)+(3+i)=4+3i$. **With $\overrightarrow{OA}=1+2i$ and $\overrightarrow{OB}=3+i$, the diagonal $\overrightarrow{OC}=4+3i$ of parallelogram $OACB$.**
> Add real parts and imaginary parts. Geometrically, slide the vector OB so its tail is at A (tip to tail): it ends at C.

**(b)** If $z=1+3i$ and $w=4+i$, plot $z$, $w$, $-z$ and $z+w$.

> $z+w=5+4i$ **and** $-z=-1-3i$.
> $-z$ is $z$ rotated by $180^\circ$ about $O$ (same length, opposite direction). $z+w$ is the diagonal of the parallelogram on $z$ and $w$.

## Example 3 · Subtraction, tip to tail

**1.** $u+v$

> $u+v=4$ **(a horizontal vector of length 4).**
> Place the tail of v at the tip of u, or take the long diagonal of the parallelogram on u and v.

**2.** $u-v$

> $u-v=u+(-v)=2+4i$**: the vector from $V$ to $U$.**
> Head minus tail: the vector pointing from the tip of v to the tip of u. It is the other diagonal of the parallelogram.

**3.** $v-u$

> $v-u=-2-4i$**: the vector from $U$ to $V$.**
> This is the opposite of the vector in part 2: same length, reversed direction.

## Example 4

$OABC$ is a parallelogram with $A$ at $5+i$ and $C$ at $2+3i$. Find the complex numbers represented by (a) $B$, (b) $\overrightarrow{AC}$, (c) $\overrightarrow{CA}$, and (d) the lengths of both diagonals.

> **(a)** $b=7+4i$**. (b)** $\overrightarrow{AC}=-3+2i$**. (c)** $\overrightarrow{CA}=3-2i$**. (d)** $OB=\sqrt{65}$ **and** $AC=\sqrt{13}$**.**
> (a) $\overrightarrow{OB}=\overrightarrow{OA}+\overrightarrow{OC}=(5+i)+(2+3i)$. (b) head minus tail: $c-a=(2+3i)-(5+i)$. (c) the opposite vector, $a-c$. (d) $|7+4i|=\sqrt{49+16}$ and $|-3+2i|=\sqrt{9+4}$.

## Example 5

Prove the triangle inequality (a) geometrically, using the parallelogram, and (b) algebraically, using $|z|^2 = z\overline{z}$.

**(a) Let $A$ represent $z_1$ and $C$ represent $z_1 + z_2$, so $AC = |z_2|$. In $\triangle OAC$, $OC \le OA + AC$ because one side of a triangle is at most the sum of the other two. That is $|z_1 + z_2| \le |z_1| + |z_2|$.**
**(b)**

$$\begin{aligned} |z_1 + z_2|^2 &= (z_1 + z_2)(\overline{z_1} + \overline{z_2}) = |z_1|^2 + |z_2|^2 + z_1\overline{z_2} + \overline{z_1}z_2 \\ &= |z_1|^2 + |z_2|^2 + 2\,\mathrm{Re}(z_1\overline{z_2}) \\ &\le |z_1|^2 + |z_2|^2 + 2|z_1\overline{z_2}| = (|z_1| + |z_2|)^2. \end{aligned}$$

**Both sides are non-negative, so take square roots.**
In (b), $z_1\overline{z_2} + \overline{z_1}z_2$ is a number plus its conjugate, so it equals $2\,\mathrm{Re}(z_1\overline{z_2})$. Then use $\mathrm{Re}(w) \le |w|$ and $|z_1\overline{z_2}| = |z_1||z_2|$. In (a), the triangle is degenerate (equality) when $O$, $A$, $C$ are collinear with $A$ between $O$ and $C$.

## Example 6(a)

Given $|z| < \dfrac{1}{2}$, show that $\left|(1+i)z^3 + iz\right| < \dfrac{3}{4}$.

$\left|(1+i)z^3 + iz\right| \le |1+i||z|^3 + |i||z| = \sqrt{2}|z|^3 + |z| < \dfrac{\sqrt{2}}{8} + \dfrac{1}{2} < \dfrac{3}{4}$.
Apply the triangle inequality, then $|z_1 z_2| = |z_1||z_2|$. Since $\frac{\sqrt{2}}{8} \approx 0.177$, the bound is about $0.677 < 0.75$.

## Example 6(b)

$z$ moves on the circle $|z| = 1$. Find the greatest value of $|z + 2| + |z - 2|$.

**The greatest value is $2\sqrt{5}$, at $z = \pm i$.**
With $z = \cos\theta + i\sin\theta$, $|z \pm 2| = \sqrt{5 \pm 4\cos\theta}$. Then $\left(\sqrt{5 + 4c} + \sqrt{5 - 4c}\right)^2 = 10 + 2\sqrt{25 - 16c^2}$, which is greatest when $c = \cos\theta = 0$, giving 20 and so $2\sqrt{5}$. The triangle inequality alone only gives the bound $6$, which is never reached: a good point to discuss.

## HSC practice · Adding unit vectors

Let $z = \dfrac{\sqrt{3}}{2} + \dfrac{1}{2}i$ and $w = -\dfrac{\sqrt{2}}{2} + \dfrac{\sqrt{2}}{2}i$. (i) Write $z$ and $w$ in polar form. (ii) Show that $|z+w|^2 = \dfrac{4+\sqrt{2}-\sqrt{6}}{2}$. (iii) $A$ and $C$ represent $z$ and $z+w$. Explain why $\angle AOC = \dfrac{7\pi}{24}$, and hence find $\cos\dfrac{7\pi}{24}$.

> **(i)** $z = \cos\frac{\pi}{6} + i\sin\frac{\pi}{6}$ **and** $w = \cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}$**. (ii)** $|z+w|^2 = 2 + 2\cos\dfrac{7\pi}{12} = 2 + \dfrac{\sqrt{2}-\sqrt{6}}{2} = \dfrac{4+\sqrt{2}-\sqrt{6}}{2}$**. (iii)** $|z| = |w| = 1$**, so** $OACB$ **is a rhombus (with** $B$ **representing** $w$**) and the diagonal** $OC$ **bisects** $\angle AOB = \frac{3\pi}{4} - \frac{\pi}{6} = \frac{7\pi}{12}$**, giving** $\angle AOC = \frac{7\pi}{24}$**. In** $\triangle OAC$ **the perpendicular from** $A$ **meets** $OC$ **at its midpoint, so** $\cos\dfrac{7\pi}{24} = \dfrac{|z+w|/2}{1} = \dfrac{\sqrt{8+2\sqrt{2}-2\sqrt{6}}}{4} \approx 0.6088$.
>
> (ii) $|z+w|^2 = |z|^2 + |w|^2 + 2\operatorname{Re}(z\overline{w})$, and $z\overline{w} = \operatorname{cis}\left(\frac{\pi}{6} - \frac{3\pi}{4}\right)$ has real part $\cos\frac{7\pi}{12} = \frac{\sqrt{2}-\sqrt{6}}{4}$. (iii) The diagonals of a rhombus bisect its angles and meet at right angles, so the foot of the perpendicular from $A$ is the midpoint of $OC$. Then $\cos\frac{7\pi}{24} = \frac{|z+w|}{2} = \frac{1}{2}\sqrt{\frac{4+\sqrt{2}-\sqrt{6}}{2}} = \frac{\sqrt{8+2\sqrt{2}-2\sqrt{6}}}{4}$. (The original paper uses the exponential form.)

## Exit ticket

**1.** $Z$ is $2+i$ and $W$ is $-1+5i$. Find the complex number represented by $\overrightarrow{WZ}$ and the length $WZ$.

> $\overrightarrow{WZ} = 3 - 4i$ **and** $WZ = 5$.
>
> Head minus tail: $z - w = (2+i) - (-1+5i) = 3 - 4i$, and $|3-4i| = \sqrt{9+16} = 5$.

**2.** Explain geometrically why $|z+w| = |z-w|$ exactly when $OZ \perp OW$.

> $z+w$ **and** $z-w$ **are the two diagonals of the parallelogram on** $OZ$ **and** $OW$**. The diagonals are equal exactly when the parallelogram is a rectangle, that is, when** $OZ \perp OW$**.**
>
> A parallelogram with equal diagonals is a rectangle, so adjacent sides are perpendicular. This idea is used again in Lesson 8.

## Practice worksheet

### Vectors and parallelograms

**1.** $P$, $Q$ and $R$ represent $1+4i$, $-2+i$ and $3-2i$. Find the complex numbers represented by (a) $\overrightarrow{PQ}$, (b) $\overrightarrow{QR}$, (c) $\overrightarrow{RP}$, and show that $\overrightarrow{PQ} + \overrightarrow{QR} + \overrightarrow{RP} = 0$.

> **(a)** $q - p = -3 - 3i$**. (b)** $r - q = 5 - 3i$**. (c)** $p - r = -2 + 6i$**. Sum:** $(-3+5-2) + (-3-3+6)i = 0$**.**

**2.** $OABC$ is a parallelogram with $A$ at $6-i$ and $C$ at $-1+4i$. Find $B$, $\overrightarrow{AC}$ and the lengths of the two diagonals.

> $B = 5 + 3i$**.** $\overrightarrow{AC} = c - a = -7 + 5i$**.** $OB = \sqrt{34}$ **and** $AC = \sqrt{74}$**.**

### Distance and direction

**3.** $W$ is $2+3i$ and $Z$ is $5+7i$. Find $|z-w|$ and sketch $\overrightarrow{WZ}$ on an Argand diagram. Find $\arg(z-w)$ correct to two decimal places.

> $z-w=3+4i$, **so** $|z-w|=5$ **and** $\arg(z-w)=\tan^{-1}\frac{4}{3}\approx 0.93$ **radians.**

### Triangle inequality

**4.** Show that for any complex numbers $|z_1-z_2|\geq\big||z_1|-|z_2|\big|$.

> $|z_1|=|(z_1-z_2)+z_2|\leq|z_1-z_2|+|z_2|$, **so** $|z_1|-|z_2|\leq|z_1-z_2|$. **Swapping the roles gives** $|z_2|-|z_1|\leq|z_1-z_2|$. **Together these give the result.**

**5.** Given $|z|\leq 2$, find the greatest possible value of $|z^2+3z|$ using the triangle inequality, and explain why it is actually attained.

> $|z^2+3z|\leq|z|^2+3|z|\leq 4+6=10$. **At** $z=2$, $|4+6|=10$, **so the bound is attained.**

**6.** Prove that $|z+w|^2+|z-w|^2=2|z|^2+2|w|^2$, and interpret it for a parallelogram.

> $|z+w|^2+|z-w|^2=(|z|^2+|w|^2+2\operatorname{Re}(z\overline{w}))+(|z|^2+|w|^2-2\operatorname{Re}(z\overline{w}))=2|z|^2+2|w|^2$. **In a parallelogram, the sum of the squares of the diagonals equals the sum of the squares of the four sides.**

### HSC-style

**7.** Adapted HSC-style. Let $z=\operatorname{cis}\dfrac{\pi}{3}$ and $w=\operatorname{cis}\dfrac{\pi}{2}$. (a) Show that $|z+w|=2\cos\dfrac{\pi}{12}$. (b) Hence find $\arg(z+w)$ and explain your answer using a rhombus. (c) By equating real parts, show that $2\cos\dfrac{\pi}{12}\cos\dfrac{5\pi}{12}=\dfrac{1}{2}$.

> **(a)** $|z+w|^2=2+2\cos\left(\frac{\pi}{3}-\frac{\pi}{2}\right)=2+2\cos\frac{\pi}{6}=2+\sqrt{3}$, **and** $\left(2\cos\frac{\pi}{12}\right)^2=2(1+\cos\frac{\pi}{6})=2+\sqrt{3}$, **so** $|z+w|=2\cos\frac{\pi}{12}$. **(b)** $|z|=|w|=1$ **gives a rhombus, so** $z+w$ **bisects the angle:** $\arg(z+w)=\frac{5\pi}{12}$. **(c)** $\operatorname{Re}(z+w)=\cos\frac{\pi}{3}+\cos\frac{\pi}{2}=\frac{1}{2}$, **and also** $\operatorname{Re}(z+w)=|z+w|\cos\arg(z+w)=2\cos\frac{\pi}{12}\cos\frac{5\pi}{12}$.