MCINTYRE MATHS · MATHEMATICS EXTENSION 2

Worked solutions

Lesson 6 · Multiplying and dividing in polar form

Warm-up · Retrieval from Lesson 5

(a) Write $1 - i$ in polar form.

> $\sqrt{2}\left(\cos\left(-\dfrac{\pi}{4}\right) + i\sin\left(-\dfrac{\pi}{4}\right)\right)$
> $r = \sqrt{2}$. The point is in the fourth quadrant with reference angle $\frac{\pi}{4}$, so $\theta = -\frac{\pi}{4}$.

(b) State $\operatorname{Arg}(-2)$.

> π **(not** $-\pi$**)**
> The principal argument lies in $(-\pi, \pi]$, which includes π and excludes $-\pi$.

(c) Expand $\cos(A+B)$ and $\sin(A+B)$.

> $\cos(A+B) = \cos A\cos B - \sin A\sin B$ **and** $\sin(A+B) = \sin A\cos B + \cos A\sin B$**.**
> Keep these on the board: they are exactly what Example 1 needs.

(d) Write $2\left(\cos\dfrac{5\pi}{6} + i\sin\dfrac{5\pi}{6}\right)$ as $x + iy$.

> $-\sqrt{3} + i$
> $2\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) = -\sqrt{3} + i.$

Example 1 · Proving the multiplication rule

Prove that

$$z_1 z_2 = r_1 r_2\big(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\big).$$

$$\begin{aligned} z_1 z_2 &= r_1 r_2(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) \\ &= r_1 r_2\big[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)\big] \\ &= r_1 r_2\big(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\big). \end{aligned}$$

Expand using $i^2 = -1$, collect real and imaginary parts, then recognise the compound-angle formulas for $\cos(\theta_1+\theta_2)$ and $\sin(\theta_1+\theta_2)$. So $|z_1 z_2| = |z_1||z_2|$ and $\arg(z_1 z_2) = \arg z_1 + \arg z_2$, up to a multiple of 2π: multiply the moduli and add the arguments.

Think

In words: what does multiplying z by $r\operatorname{cis}\theta$ do to the point z on the Argand diagram?

> **It multiplies the distance from O by r (a dilation about O) and adds θ to the argument (a rotation about O through θ).**
> This is the geometric meaning of "multiply the moduli, add the arguments". It is developed fully in Lesson 8.

Example 2 · Multiplying in polar form

(a) $4\operatorname{cis}\dfrac{3\pi}{4}\times 3\operatorname{cis}\dfrac{\pi}{2}$

> $12\operatorname{cis}\left(-\dfrac{3\pi}{4}\right)$
> Multiply moduli: $4\times 3=12$. Add arguments: $\frac{3\pi}{4}+\frac{\pi}{2}=\frac{5\pi}{4}$. This is more than π, so subtract 2π to get $-\frac{3\pi}{4}$.

(b) $\left(2\operatorname{cis}\dfrac{2\pi}{3}\right)^2$

> $4\operatorname{cis}\left(-\dfrac{2\pi}{3}\right)$
> $2^2=4$ and $\frac{2\pi}{3}+\frac{2\pi}{3}=\frac{4\pi}{3}$, which is outside $(-\pi,\pi]$, so subtract 2π to get $-\frac{2\pi}{3}$.

Example 3 · Dividing in polar form

(a) $\dfrac{12\operatorname{cis}0}{6\operatorname{cis}\frac{2\pi}{3}}$

> $2\operatorname{cis}\left(-\dfrac{2\pi}{3}\right)=2\left(\cos\dfrac{2\pi}{3}-i\sin\dfrac{2\pi}{3}\right)=-1-\sqrt{3}\,i$
> Divide moduli: $12\div 6=2$. Subtract arguments: $0-\frac{2\pi}{3}=-\frac{2\pi}{3}$, already in $(-\pi,\pi]$.

(b) $\dfrac{6\operatorname{cis}\left(-\frac{\pi}{3}\right)}{2\operatorname{cis}\frac{2\pi}{3}}$

> $3\operatorname{cis}\pi=-3$
> $6\div 2=3$ and $-\frac{\pi}{3}-\frac{2\pi}{3}=-\pi$. The principal argument lies in $(-\pi,\pi]$, so write π instead of $-\pi$. A negative real number has principal argument π.

Example 4 · Proving the identities

(a) Prove that $\left|\dfrac{z_1}{z_2}\right|=\dfrac{|z_1|}{|z_2|}$ and $\arg\left(\dfrac{z_1}{z_2}\right)=\arg z_1-\arg z_2$.

> **Let $w=\dfrac{z_1}{z_2}$, so $z_1=wz_2$. By the product rule, $|z_1|=|w||z_2|$ and $\arg z_1=\arg w+\arg z_2$.**
> **Rearranging gives $|w|=\dfrac{|z_1|}{|z_2|}$ and $\arg w=\arg z_1-\arg z_2$.**
> Use the product rule you have already proved, applied to $z_1=wz_2$. (Alternatively, multiply top and bottom by $\cos\theta_2-i\sin\theta_2$ and simplify.)

(b) Prove that $\arg \overline{z} = -\arg z$ and $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$ using polar form.

> **If $z = r\operatorname{cis}\theta$ then $\overline{z} = r(\cos\theta - i\sin\theta) = r(\cos(-\theta) + i\sin(-\theta))$, so $\arg\overline{z} = -\theta$. Then $\overline{z_1}\,\overline{z_2} = r_1\operatorname{cis}(-\theta_1)\cdot r_2\operatorname{cis}(-\theta_2) = r_1 r_2\operatorname{cis}(-(\theta_1+\theta_2)) = \overline{z_1 z_2}$.**
> Cosine is even and sine is odd. The conjugate of a product has modulus $r_1 r_2$ and argument $-(\theta_1 + \theta_2)$, which is exactly the product of the conjugates.

Example 5 · Two forms, exact values

(a) Write z_1 and z_2 in polar form.

> **$z_1 = 2\operatorname{cis}\dfrac{\pi}{3}$ and $z_2 = \sqrt{2}\operatorname{cis}\dfrac{\pi}{4}$.**
> $|z_1| = \sqrt{1+3} = 2$ with $\tan\theta = \sqrt{3}$. $|z_2| = \sqrt{2}$ with $\tan\theta = 1$. Both are in the first quadrant.

(b) Find $z_1 z_2$ and $\dfrac{z_1}{z_2}$ in polar form and in Cartesian form.

> **$z_1 z_2 = 2\sqrt{2}\operatorname{cis}\dfrac{7\pi}{12} = (1-\sqrt{3}) + (1+\sqrt{3})i$. $\dfrac{z_1}{z_2} = \sqrt{2}\operatorname{cis}\dfrac{\pi}{12} = \dfrac{(1+\sqrt{3}) + (\sqrt{3}-1)i}{2}$.**
> Polar: multiply moduli and add arguments ($\frac{\pi}{3} + \frac{\pi}{4} = \frac{7\pi}{12}$), or divide moduli and subtract ($\frac{\pi}{3} - \frac{\pi}{4} = \frac{\pi}{12}$). Cartesian: expand $(1+\sqrt{3}i)(1+i)$, and for the quotient multiply top and bottom by $1-i$.

(c) Hence find the exact values of $\sin\dfrac{7\pi}{12}$ and $\cos\dfrac{\pi}{12}$.

> **$\sin\dfrac{7\pi}{12} = \cos\dfrac{\pi}{12} = \dfrac{\sqrt{2}+\sqrt{6}}{4}$.**
> Equate imaginary parts of the two forms of $z_1 z_2$: $2\sqrt{2}\sin\frac{7\pi}{12} = 1+\sqrt{3}$, so $\sin\frac{7\pi}{12} = \frac{1+\sqrt{3}}{2\sqrt{2}} = \frac{\sqrt{2}+\sqrt{6}}{4}$. Equate real parts of $\frac{z_1}{z_2}$: $\sqrt{2}\cos\frac{\pi}{12} = \frac{1+\sqrt{3}}{2}$, so $\cos\frac{\pi}{12} = \frac{\sqrt{2}+\sqrt{6}}{4}$.

HSC practice · Powers in polar form

Let $z = \sqrt{3} + i$. (i) Write z in polar form. (ii) Hence express z^7 in the form $x + iy$.

> **(i) $z = 2\operatorname{cis}\dfrac{\pi}{6}$. (ii) $z^7 = 128\operatorname{cis}\dfrac{7\pi}{6} = -64\sqrt{3} - 64i$.**
> $\arg(z^n) = n\arg z$ and $|z^n| = |z|^n$, so $z^7 = 2^7\operatorname{cis}\frac{7\pi}{6} = 128\left(-\frac{\sqrt{3}}{2} - \frac{1}{2}i\right)$.

HSC practice · Powers in polar form

Let $z = -\sqrt{3} + i$. Use polar form to find z^{10} in the form $x + iy$.

> **$z = 2\operatorname{cis}\dfrac{5\pi}{6}$ and $z^{10} = 1024\operatorname{cis}\dfrac{25\pi}{3} = 1024\operatorname{cis}\dfrac{\pi}{3} = 512 + 512\sqrt{3}\,i$.**
> $10\times\frac{5\pi}{6} = \frac{25\pi}{3} = 8\pi + \frac{\pi}{3}$. Remove whole turns of 2π, then use $\cos\frac{\pi}{3} = \frac{1}{2}$ and $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$. The rule $\arg(z^n) = n\arg z$ is enough here; de Moivre's theorem is proved by induction later in this unit.

HSC practice · Modulus and argument of a new number

Let z be a non-zero complex number and $w = \dfrac{z^2}{|z|}$. Find $|w|$ and $\arg w$ in terms of $|z|$ and $\arg z$. Describe where w lies compared with z.

> **$|w| = |z|$ and $\arg w = 2\arg z$. So w is on the same circle $|w| = |z|$ about O as z, at double the angle.**
> $|w| = \dfrac{|z|^2}{|z|} = |z|$. Dividing by the positive real number $|z|$ does not change the argument (a positive real has argument 0), so $\arg w = \arg(z^2) - 0 = 2\arg z$.

Exit ticket

1. Use polar form to express $\dfrac{(1+i)^5}{(1-i)^3}$ in the form $x + iy$.

> 2
> $1 + i = \sqrt{2}\,\mathrm{cis}\,\frac{\pi}{4}$ and $1 - i = \sqrt{2}\,\mathrm{cis}\left(-\frac{\pi}{4}\right)$. The quotient is $\dfrac{(\sqrt{2})^5\,\mathrm{cis}\,\frac{5\pi}{4}}{(\sqrt{2})^3\,\mathrm{cis}\left(-\frac{3\pi}{4}\right)} =$
> $2\,\mathrm{cis}\left(\frac{5\pi}{4} + \frac{3\pi}{4}\right) = 2\,\mathrm{cis}\,2\pi = 2$.

2. Prove that $|z^n| = |z|^n$ and $\arg(z^n) = n\arg z$ for $n = 2$ and $n = 3$.

> **With $z = r\,\mathrm{cis}\,\theta$: $z^2 = z \cdot z = r^2\,\mathrm{cis}\,2\theta$, and $z^3 = z^2 \cdot z = r^3\,\mathrm{cis}\,3\theta$.**
> Apply the product rule once for $n = 2$ and again for $n = 3$. The general case is proved by induction in Complex Numbers II (de Moivre's theorem).

Practice worksheet

Multiplying and dividing

1. Express in polar form with the principal argument.

(a) $3\,\mathrm{cis}\,\frac{2\pi}{3} \times 2\,\mathrm{cis}\,\frac{3\pi}{4}$ (b) $\dfrac{8\,\mathrm{cis}\,\frac{\pi}{6}}{2\,\mathrm{cis}\,\frac{5\pi}{6}}$

(c) $\left(\sqrt{2}\,\mathrm{cis}\,\frac{3\pi}{4}\right)^3$

> **(a)** $6\,\mathrm{cis}\,\frac{17\pi}{12} = 6\,\mathrm{cis}\left(-\frac{7\pi}{12}\right)$. **(b)** $4\,\mathrm{cis}\left(-\frac{2\pi}{3}\right)$. **(c)** $2\sqrt{2}\,\mathrm{cis}\,\frac{9\pi}{4} = 2\sqrt{2}\,\mathrm{cis}\,\frac{\pi}{4}$.

2. Write $z_1 = \sqrt{3} - i$ and $z_2 = -1 + i$ in polar form, then find $z_1 z_2$ and $\dfrac{z_1}{z_2}$ in polar form.

> $z_1 = 2\,\mathrm{cis}\left(-\frac{\pi}{6}\right)$, $z_2 = \sqrt{2}\,\mathrm{cis}\,\frac{3\pi}{4}$. $z_1 z_2 = 2\sqrt{2}\,\mathrm{cis}\,\frac{7\pi}{12}$. $\dfrac{z_1}{z_2} = \sqrt{2}\,\mathrm{cis}\left(-\frac{\pi}{6} - \frac{3\pi}{4}\right) = \sqrt{2}\,\mathrm{cis}\left(-\frac{11\pi}{12}\right)$.

Exact values

3. Let $z_1 = \sqrt{3} + i$ and $z_2 = 1 - i$. Find $z_1 z_2$ in polar form and in Cartesian form. Hence find the exact value of $\cos\left(-\frac{\pi}{12}\right)$.

> $z_1 = 2\,\mathrm{cis}\,\frac{\pi}{6}$, $z_2 = \sqrt{2}\,\mathrm{cis}\left(-\frac{\pi}{4}\right)$, **so** $z_1 z_2 = 2\sqrt{2}\,\mathrm{cis}\left(-\frac{\pi}{12}\right)$. **Cartesian:** $(\sqrt{3}+i)(1-i) = (\sqrt{3}+1) + (1-\sqrt{3})i$. **Equating real parts,** $2\sqrt{2}\cos\frac{\pi}{12} = \sqrt{3}+1$, **so** $\cos\frac{\pi}{12} = \frac{\sqrt{6}+\sqrt{2}}{4}$.

Proofs

4. Prove that $\arg\left(\dfrac{1}{z}\right) = -\arg z$ for $z \neq 0$, and explain the connection with the conjugate $\overline{z}$.

> **With $1 = 1\operatorname{cis}0$ and $z = r\operatorname{cis}\theta$, the quotient rule gives $\frac{1}{z} = \frac{1}{r}\operatorname{cis}(0-\theta)$, so $\arg\frac{1}{z} = -\theta$. Since $\overline{z} = r\operatorname{cis}(-\theta)$ has the same argument, $\frac{1}{z}$ and $\overline{z}$ point in the same direction, consistent with $\frac{1}{z} = \frac{\overline{z}}{|z|^2}$.**

5. Prove that $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\overline{z_1}}{\overline{z_2}}$ using polar form.

> **$\overline{z_1} = r_1\operatorname{cis}(-\theta_1)$ and $\overline{z_2} = r_2\operatorname{cis}(-\theta_2)$, so $\dfrac{\overline{z_1}}{\overline{z_2}} = \dfrac{r_1}{r_2}\operatorname{cis}(-\theta_1+\theta_2)$. Also $\dfrac{z_1}{z_2} = \dfrac{r_1}{r_2}\operatorname{cis}(\theta_1-\theta_2)$, whose conjugate is $\dfrac{r_1}{r_2}\operatorname{cis}(-(\theta_1-\theta_2))$, the same.**

Powers

6. Use polar form to find $(1-i)^8$ and $\left(-1+\sqrt{3}\,i\right)^6$.

> **$1-i = \sqrt{2}\operatorname{cis}(-\frac{\pi}{4})$, so $(1-i)^8 = 16\operatorname{cis}(-2\pi) = 16$. $-1+\sqrt{3}i = 2\operatorname{cis}\frac{2\pi}{3}$, so the sixth power is $64\operatorname{cis}4\pi = 64$.**

HSC-style

7. Let $z = 1+i$ and $w = \sqrt{3}+i$. (a) Write z and w in polar form. (b) Find $\dfrac{w^6}{z^4}$. (c) Find the smallest positive integer n such that $\left(\dfrac{w}{z}\right)^n$ is real.

> **(a) $z = \sqrt{2}\operatorname{cis}\frac{\pi}{4}$, $w = 2\operatorname{cis}\frac{\pi}{6}$. (b) $w^6 = 64\operatorname{cis}\pi$ and $z^4 = 4\operatorname{cis}\pi$, so $\frac{w^6}{z^4} = 16\operatorname{cis}0 = 16$. (c) $\frac{w}{z} = \sqrt{2}\operatorname{cis}\left(-\frac{\pi}{12}\right)$, so $\left(\frac{w}{z}\right)^n$ has argument $-\frac{n\pi}{12}$, which is a multiple of π first when $n = 12$.**