MCINTYRE MATHS · MATHEMATICS EXTENSION 2

Worked solutions

Lesson 5 · Modulus–argument form

Warm-up · Exact values

(a) $\tan^{-1}\sqrt{3}$

> $\dfrac{\pi}{3}$
> $\tan\frac{\pi}{3}=\sqrt{3}$, and $\frac{\pi}{3}$ lies in $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$.

(b) $\cos\dfrac{2\pi}{3}$

> $-\dfrac{1}{2}$
> $\frac{2\pi}{3}$ is in the second quadrant, where cosine is negative, with reference angle $\frac{\pi}{3}$.

(c) $\sin\left(-\dfrac{3\pi}{4}\right)$

> $-\dfrac{1}{\sqrt{2}}$
> $-\frac{3\pi}{4}$ is in the third quadrant, where sine is negative, with reference angle $\frac{\pi}{4}$.

(d) Find $\theta\in(-\pi,\pi]$ with $\cos\theta=-\dfrac{1}{2}$ and $\sin\theta=-\dfrac{\sqrt{3}}{2}$.

> $\theta=-\dfrac{2\pi}{3}$
> Both cosine and sine are negative, so θ is in the third quadrant. The reference angle is $\frac{\pi}{3}$, so $\theta=-\left(\pi-\frac{\pi}{3}\right)=-\frac{2\pi}{3}$.

Discuss

(a) Sketch: mark r and θ for a point $z=x+iy$ in each quadrant.

> **In each quadrant, r is the length of the line from O to z and θ is the angle from the positive real axis to that line, anticlockwise positive and clockwise negative. For example, $3+4i$ has $r=5$ and θ marked above the axis, while $3-4i$ has the same r and $-\theta$ below it.**
> The sketch shows why the quadrant matters: $\cos\theta=\frac{x}{r}$ and $\sin\theta=\frac{y}{r}$ have signs that depend on where the point is.

(b) Write three different values of $\arg(1+i)$. Which one is $\operatorname{Arg}(1+i)$?

> **For example $\frac{\pi}{4}$, $\frac{9\pi}{4}$ and $-\frac{7\pi}{4}$ (any $\frac{\pi}{4}+2k\pi$). $\operatorname{Arg}(1+i)=\frac{\pi}{4}$.**
> Adding or subtracting 2π gives the same point, so arg has many values. Only $\frac{\pi}{4}$ lies in $(-\pi,\pi]$.

Example 1 · Cartesian to polar

Plot $z = 3 + 4i$. Find $|z|$ and $\operatorname{Arg} z$, and write z in modulus–argument form. Repeat for $\overline{z} = 3 - 4i$.

> **$|z| = 5$ and $\operatorname{Arg} z = \tan^{-1}\frac{4}{3} \approx 0.93$, so $z \approx 5(\cos 0.93 + i \sin 0.93)$. Also $\overline{z} \approx 5(\cos(-0.93) + i\sin(-0.93))$.**
> $r = \sqrt{3^2 + 4^2} = 5$. The point is in the first quadrant, so $\theta = \tan^{-1}\frac{4}{3} \approx 0.93$ radians (about $53°$). The conjugate is the reflection in the real axis, so it has the same modulus and the opposite argument.

Example 2 · Second and third quadrants

(a) $-1 + i$

> $\sqrt{2}\left(\cos\dfrac{3\pi}{4} + i\sin\dfrac{3\pi}{4}\right)$
> $r = \sqrt{2}$. The point is in the second quadrant with reference angle $\frac{\pi}{4}$, so $\theta = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$.

(b) $-1 - \sqrt{3}\,i$

> $2\left(\cos\left(-\dfrac{2\pi}{3}\right) + i\sin\left(-\dfrac{2\pi}{3}\right)\right)$
> $r = \sqrt{1+3} = 2$. The point is in the third quadrant with reference angle $\frac{\pi}{3}$, so $\theta = -\left(\pi - \frac{\pi}{3}\right) = -\frac{2\pi}{3}$.

Example 3 · Choosing the quadrant

(a) Find $\operatorname{Arg}\left(\sqrt{6} - \sqrt{2}\,i\right)$.

> $-\dfrac{\pi}{6}$
> The point is in the fourth quadrant. $\tan\alpha = \frac{\sqrt{2}}{\sqrt{6}} = \frac{1}{\sqrt{3}}$, so $\alpha = \frac{\pi}{6}$ and $\operatorname{Arg} = -\frac{\pi}{6}$. (The modulus is $\sqrt{8} = 2\sqrt{2}$.)

Example 3 · Choosing the quadrant

(b) Write $-4 - 3i$ in polar form, with the argument correct to 2 decimal places.

> $5\left(\cos(-2.50) + i\sin(-2.50)\right)$
> $r = 5$ (a 3-4-5 triangle). The point is in the third quadrant, with $\alpha = \tan^{-1}\frac{3}{4} \approx 0.64$. So $\operatorname{Arg} = -\pi + 0.64 \approx -2.50$.

Example 4 · On the axes

(a) $5i$

> $5\left(\cos\dfrac{\pi}{2} + i\sin\dfrac{\pi}{2}\right)$
> The point is on the positive imaginary axis, a quarter-turn from the real axis.

(b) -4

$4(\cos\pi + i\sin\pi)$

$\operatorname{Arg}(-4) = \pi$, not $-\pi$: the interval $(-\pi, \pi]$ includes π and excludes $-\pi$.

(c) $-\dfrac{i}{2}$

$\dfrac{1}{2}\left(\cos\left(-\dfrac{\pi}{2}\right) + i\sin\left(-\dfrac{\pi}{2}\right)\right)$

The point is on the negative imaginary axis, a quarter-turn clockwise from the real axis.

Example 5 · Polar to Cartesian

(a) $4\left(\cos\dfrac{\pi}{3} + i\sin\dfrac{\pi}{3}\right)$

$2 + 2\sqrt{3}\,i$

$4\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$.

(b) $10\ \operatorname{cis}\dfrac{2\pi}{3}$

$-5 + 5\sqrt{3}\,i$

$10\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$.

(c) $\sqrt{6}\ \operatorname{cis}\left(-\dfrac{3\pi}{4}\right)$

$-\sqrt{3} - \sqrt{3}\,i$

$\sqrt{6}\left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right) = -\sqrt{3} - \sqrt{3}\,i.$

(d) $2\ \operatorname{cis}\pi$

-2

$2(-1 + 0i) = -2.$

Example 6 · Conjugate and negative in polar form

(a) $\overline{z}$

$2\ \operatorname{cis}\left(-\dfrac{\pi}{6}\right)$

Negate the argument.

(b) $-z$

$2\ \operatorname{cis}\left(-\dfrac{5\pi}{6}\right)$

$\frac{\pi}{6} - \pi = -\frac{5\pi}{6}$, which lies in $(-\pi, \pi]$.

(c) $-\overline{z}$

$2\ \operatorname{cis}\dfrac{5\pi}{6}$

$-\frac{\pi}{6} + \pi = \frac{5\pi}{6}$.

(d) Plot all four points.

> **$z = \sqrt{3} + i$, $\overline{z} = \sqrt{3} - i$, $-z = -\sqrt{3} - i$ and $-\overline{z} = -\sqrt{3} + i$: the vertices of a rectangle centred at the origin.**
> All four have modulus 2, so they lie on a circle of radius 2.

HSC practice · Polar form

Let $z = -\sqrt{3} + i$. (i) Find $|z|$ and $\operatorname{Arg} z$. (ii) Write z in the form $r(\cos\theta + i\sin\theta)$.

> **(i) $|z| = 2$ and $\operatorname{Arg} z = \dfrac{5\pi}{6}$. (ii) $z = 2\left(\cos\dfrac{5\pi}{6} + i\sin\dfrac{5\pi}{6}\right)$.**
> $|z| = \sqrt{3+1} = 2$. The point is in the second quadrant with $\tan\alpha = \frac{1}{\sqrt{3}}$, so $\alpha = \frac{\pi}{6}$ and $\operatorname{Arg} z = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$.

Spot the error · Arguments

(a) $z = -2 + 2i$: $\operatorname{Arg} z = \tan^{-1}\left(\dfrac{2}{-2}\right) = -\dfrac{\pi}{4}$

> $\operatorname{Arg} z = \dfrac{3\pi}{4}$
> The point $-2 + 2i$ is in the second quadrant, but $\tan^{-1}$ only returns values in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. The reference angle is $\frac{\pi}{4}$, so $\operatorname{Arg} z = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$. This is why the method says to sketch first.

(b) $\operatorname{Arg}(-4) = -\pi$

> $\operatorname{Arg}(-4) = \pi$
> The principal argument lies in $(-\pi, \pi]$, which includes π and excludes $-\pi$.

(c) $1 - \sqrt{3}\,i = 2\left(\cos\dfrac{\pi}{3} + i\sin\dfrac{\pi}{3}\right)$

> $1 - \sqrt{3}\,i = 2\left(\cos\left(-\dfrac{\pi}{3}\right) + i\sin\left(-\dfrac{\pi}{3}\right)\right)$
> The point is in the fourth quadrant (positive real part, negative imaginary part), so the argument is $-\frac{\pi}{3}$. The written form $2\operatorname{cis}\frac{\pi}{3}$ is the point $1 + \sqrt{3}\,i$.

(d) $2\left(\cos\dfrac{\pi}{6} - i\sin\dfrac{\pi}{6}\right)$ has argument $\dfrac{\pi}{6}$

> **The argument is $-\dfrac{\pi}{6}$.**
> The expression is not in polar form because of the minus sign. Since cosine is even and sine is odd, $\cos\frac{\pi}{6} - i\sin\frac{\pi}{6} = \cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)$.

Exit ticket

1. Write $-2\sqrt{3} + 2i$ in polar form.

> $4\left(\cos\dfrac{5\pi}{6} + i\sin\dfrac{5\pi}{6}\right)$
> $r = \sqrt{12+4} = 4$. The point is in the second quadrant, with $\alpha = \tan^{-1}\frac{2}{2\sqrt{3}} = \frac{\pi}{6}$, so $\theta = \frac{5\pi}{6}$.

2. Express $6\left(\cos\left(-\frac{\pi}{4}\right)+i\sin\left(-\frac{\pi}{4}\right)\right)$ as $x+iy$.

> $3\sqrt{2}-3\sqrt{2}\,i$
> $6\left(\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}i\right)=\frac{6}{\sqrt{2}}-\frac{6}{\sqrt{2}}i$ and $\frac{6}{\sqrt{2}}=3\sqrt{2}$.

3. Explain why $\operatorname{Arg}(-5)=\pi$ and not $-\pi$.

> **The principal argument lies in $(-\pi,\pi]$: π is included and $-\pi$ is not.**
> $\arg(-5)$ has the values $\pi+2k\pi$, including both π and $-\pi$. Exactly one value must be chosen as Arg, and the interval $(-\pi,\pi]$ is closed at π and open at $-\pi$.

Practice worksheet

Cartesian to polar

1. Write in modulus–argument form with the principal argument in exact form.

(a) $1+i$ (b) $-\sqrt{3}+i$

(c) $-2-2i$ (d) $\sqrt{3}-3i$

> **(a)** $\sqrt{2}\operatorname{cis}\frac{\pi}{4}$. **(b)** $2\operatorname{cis}\frac{5\pi}{6}$. **(c)** $r=2\sqrt{2}$, **third quadrant:** $2\sqrt{2}\operatorname{cis}\left(-\frac{3\pi}{4}\right)$. **(d)** $r=\sqrt{3+9}=2\sqrt{3}$, **fourth quadrant with** $\tan\alpha=\sqrt{3}$**:** $2\sqrt{3}\operatorname{cis}\left(-\frac{\pi}{3}\right)$.

2. Write in modulus–argument form.

(a) -7 (b) $-3i$

(c) 12

> **(a)** $7(\cos\pi+i\sin\pi)$. **(b)** $3\left(\cos\left(-\frac{\pi}{2}\right)+i\sin\left(-\frac{\pi}{2}\right)\right)$. **(c)** $12(\cos 0+i\sin 0)$.

3. Write $-5+12i$ in polar form with the argument correct to 2 decimal places.

> $r=13$. **Second quadrant:** $\operatorname{Arg}=\pi-\tan^{-1}\frac{12}{5}\approx 3.1416-1.1760=1.97$. **So** $z\approx 13(\cos 1.97+i\sin 1.97)$.

Polar to Cartesian

4. Express in the form $x+iy$.

(a) $2\operatorname{cis}\frac{\pi}{6}$ (b) $8\operatorname{cis}\left(-\frac{\pi}{2}\right)$

(c) $\sqrt{8}\operatorname{cis}\frac{3\pi}{4}$ (d) $4\operatorname{cis}\left(-\frac{2\pi}{3}\right)$

> **(a)** $\sqrt{3}+i$. **(b)** $-8i$. **(c)** $2\sqrt{2}\left(-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i\right)=-2+2i$. **(d)** $4\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)=-2-2\sqrt{3}\,i$.

Conjugate and negative

5. Let $z=3\operatorname{cis}\frac{3\pi}{4}$. Write $\overline{z}$, $-z$ and $-\overline{z}$ in polar form.

> $\overline{z}=3\operatorname{cis}\left(-\frac{3\pi}{4}\right)$. $-z$**:** $\frac{3\pi}{4}-\pi=-\frac{\pi}{4}$**, so** $-z=3\operatorname{cis}\left(-\frac{\pi}{4}\right)$. $-\overline{z}$**:** $-\frac{3\pi}{4}+\pi=\frac{\pi}{4}$**, so** $-\overline{z}=3\operatorname{cis}\frac{\pi}{4}$.

6. Explain why $\operatorname{Arg}(-5)=\pi$ and not $-\pi$. What is $\operatorname{Arg}\left(\overline{-5}\right)$?

> **The principal argument lies in $(-\pi,\pi]$: π is included and $-\pi$ is excluded. Since -5 is real, $\overline{-5}=-5$, so $\operatorname{Arg}=\pi$ again. (The rule $\operatorname{Arg}\overline{z}=-\operatorname{Arg}z$ fails for negative real numbers.)**

HSC-style

7. Let $z = 1 - i\sqrt{3}$. (i) Find $|z|$ and $\operatorname{Arg} z$. (ii) Write z in the form $r(\cos\theta + i\sin\theta)$. (iii) Write $-z$ in polar form with the principal argument.

> **(i)** $|z| = \sqrt{1+3} = 2$**. The point is in the fourth quadrant with** $\tan\alpha = \sqrt{3}$**, so** $\operatorname{Arg} z = -\frac{\pi}{3}$**. (ii)** $z = 2\left(\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right)$**. (iii)** $-z = 2\ \operatorname{cis}\left(-\frac{\pi}{3} + \pi\right) = 2\ \operatorname{cis}\frac{2\pi}{3}$**.**