MCINTYRE MATHS · MATHEMATICS EXTENSION 2

Worked solutions

Lesson 4 · The Argand diagram

Warm-up · Retrieval from Lessons 1 to 3

(a) Find real a, b with $(a+ib)(1+i) = 3+i$.

$a = 2, b = -1$

$a+ib = \dfrac{3+i}{1+i} = \dfrac{(3+i)(1-i)}{2} = \dfrac{4-2i}{2} = 2-i$. Check: $(2-i)(1+i) = 2+2i-i+1 = 3+i$.

(b) Find the square roots of $2i$.

$\pm(1+i)$

$(1+i)^2 = 1+2i-1 = 2i$, so the square roots are $\pm(1+i)$.

(c) Find a real quadratic with root $1+2i$.

$z^2 - 2z + 5 = 0$

The other root is $1-2i$. Sum $= 2$, product $= 1+4 = 5$.

(d) Find the distance between $(1, 2)$ and $(4, 6)$.

5

$\sqrt{3^2+4^2} = 5$. This previews distance on the Argand diagram.

Check · Reading the plane

(a) Write the point for $-3+5i$.

$(-3, 5)$.

The real part is the x-coordinate and the imaginary part is the y-coordinate.

(b) Write the point for $2i$.

$(0, 2)$.

$2i = 0+2i$, so it lies on the imaginary axis.

(c) Write the point for 7.

$(7, 0)$.

$7 = 7+0i$, so it lies on the real axis.

(d) Write the point for $(2+i)\,i$.

$(-1, 2)$.

$(2+i)i = 2i+i^2 = -1+2i$. This previews multiplying by i.

(e) Write the number for $(-3, 0)$.

> -3.
> A point on the real axis is a real number.

(f) Write the number for $(0, 3)$.

> $3i$.
> A point on the imaginary axis is purely imaginary.

(g) Write the number for $(7, -5)$.

> $7 - 5i$.
> $x = 7$ and $y = -5$, so $z = 7 + (-5)i$.

(h) Write the number for (a, b).

> $a + ib$.
> This is the general rule: (x, y) represents $x + iy$.

Example 1

Let $z = 3 + 4i$. Plot (a) z, (b) $\overline{z}$, (c) $-z$, (d) iz. Describe each as a transformation of z.

> $z = (3, 4)$; $\overline{z} = 3 - 4i$ **at** $(3, -4)$; $-z = -3 - 4i$ **at** $(-3, -4)$; $iz = -4 + 3i$ **at** $(-4, 3)$.
> $\overline{z}$ is the reflection of z in the real axis. $-z$ is a rotation of $180°$ about the origin. $iz = i(3 + 4i) = -4 + 3i$ is a rotation of $90°$ anticlockwise about the origin. In general $iz = i(x + iy) = -y + ix$. The gradient of Oz is $\frac{4}{3}$ and the gradient of $O(iz)$ is $-\frac{3}{4}$: the product is -1, so they are perpendicular.

Example 2 · Powers of i on the plane

(a) Plot the four points.

> $A = 1 + 3i$, $B = -3 + i$, $C = -1 - 3i$, $D = 3 - i$.
> $iz = i + 3i^2 = -3 + i$, $i^2z = -z = -1 - 3i$, $i^3z = -iz = 3 - i$.

(b) What type of quadrilateral is $ABCD$?

> **A square centred at the origin.**
> All sides have length $\sqrt{20}$ (for example AB: $(4, 2)$ has length $\sqrt{20}$) and the diagonals AC and BD are equal, both of length $2\sqrt{10}$.

(c) What does multiplying by i appear to do?

> **It rotates the point $90°$ anticlockwise about the origin.**
> Each step $A \to B \to C \to D \to A$ is a quarter-turn. Four quarter-turns return to the start, which is why $i^4 = 1$.

Example 3 · Why multiplying by i is a rotation

(a) By considering gradients, show that $OA \perp OB$.

> **$w = i(a+ib) = -b + ia$, so $A = (a, b)$ and $B = (-b, a)$. The gradient of OA is $\frac{b}{a}$ and of OB is $\frac{a}{-b}$. The product is $\frac{b}{a} \times \left(-\frac{a}{b}\right) = -1$, so $OA \perp OB$.**
> Two lines are perpendicular when the product of their gradients is -1. (If a or b is 0, the point lies on an axis and the result is clear from a sketch.)

(b) Use the distance formula to show that $OA = OB$. What type of triangle is $\triangle OAB$?

> **$OA = \sqrt{a^2 + b^2}$ and $OB = \sqrt{b^2 + a^2}$, so $OA = OB$. The triangle OAB is right-angled and isosceles.**
> So B is A rotated 90° anticlockwise about O. This is the geometric fact behind rotations in Lesson 8.

Example 4 · Lines and regions

(a) Sketch $\operatorname{Re}(z) = 5$ and $\operatorname{Im}(z) = -2$.

> **A vertical line through $(5, 0)$ and a horizontal line through $(0, -2)$.**
> They meet at the point $5 - 2i$.

(b) Shade the region where $\operatorname{Im}(z) < 1$ and $\operatorname{Re}(z) \geq -2$.

> **The region below the dashed line $y = 1$ and to the right of the solid line $x = -2$. The corner $(-2, 1)$ is not included.**
> Both conditions must hold, so shade the overlap. The corner is excluded because $\operatorname{Im}(z) < 1$ is strict there.

Example 5

Given that $z\overline{z} = 4$, find the locus of z and sketch it.

> **The circle with centre O and radius 2: $x^2 + y^2 = 4$.**
> $(x + iy)(x - iy) = x^2 + y^2 = 4$. Since $z\overline{z} = |z|^2$, this is the same as $|z| = 2$.

Example 6 · A locus with fractions

Given that $\dfrac{1}{z} + \dfrac{1}{\overline{z}} = 1$, find the locus of z and sketch it.

> **The circle $(x-1)^2 + y^2 = 1$, centre $(1, 0)$ and radius 1, excluding the origin.**
> The left side is undefined at $z = 0$, so $z \neq 0$. Multiply by $z\overline{z}$: $\overline{z} + z = z\overline{z}$, so $2x = x^2 + y^2$ and $(x-1)^2 + y^2 = 1$. Draw an open circle at the origin. (Alternatively, $\frac{1}{z} + \frac{1}{\overline{z}} = 2\operatorname{Re}\left(\frac{1}{z}\right)$ by Lesson 2.)

Example 7 · Show that it is a circle

Show that $(z - 2)(\overline{z} - 2) = 9$ represents a circle in the Argand diagram, and sketch it.

> **It is the circle with centre $(2, 0)$ and radius 3.**
> Let $z = x + iy$: $(x - 2 + iy)(x - 2 - iy) = (x-2)^2 + y^2 = 9$. Neater: $\overline{z} - 2 = \overline{z - 2}$, so the left side is $|z - 2|^2$ and the equation says $|z - 2| = 3$: the distance from z to 2 is 3.

Exam practice · Multiple choice and the unit circle

Which equation describes the circle with centre O and radius 2?

(A) $z\overline{z} = 2$ (B) $(z+\overline{z})^2 - (z-\overline{z})^2 = 16$

(C) $\mathrm{Re}(z) + \mathrm{Im}(z) = 2$ (D) $|z|^2 = 2$

> **(B)**
> $(z+\overline{z})^2 - (z-\overline{z})^2 = (2x)^2 - (2iy)^2 = 4x^2 + 4y^2 = 16$, so $x^2 + y^2 = 4$. (A) and (D) both give radius $\sqrt{2}$. (C) is a straight line.

Exam practice · Multiple choice and the unit circle

The point Z lies on the unit circle in the first quadrant, as shown. Where does $W = \dfrac{1}{Z}$ lie?

> **On the unit circle in the fourth quadrant: W is the reflection of Z in the real axis.**
> $|Z| = 1$, so $Z\overline{Z} = 1$ and $\frac{1}{Z} = \overline{Z}$. This uses $\frac{1}{z} = \frac{\overline{z}}{|z|^2}$ from Lesson 2.

Exit ticket

1. Let $z = -2 + i$. Plot z, $\overline{z}$ and iz on one Argand diagram.

> $z = (-2, 1)$, $\overline{z} = (-2, -1)$ **and** $iz = (-1, -2)$.
> $iz = i(-2+i) = -2i + i^2 = -1 - 2i$, which is z rotated 90° anticlockwise about the origin.

2. Find and sketch the locus of z if $\mathrm{Im}(z^2) = 0$.

> **Both axes: the lines $x = 0$ and $y = 0$ (the origin is included).**
> $z^2 = x^2 - y^2 + 2ixy$, so $\mathrm{Im}(z^2) = 2xy = 0$, giving $x = 0$ or $y = 0$.

Practice worksheet

Plotting and transformations

1. Let $z = -4 + 2i$. Plot z, $\overline{z}$, $-z$ and iz on one Argand diagram and describe each transformation.

> $z = (-4, 2)$; $\overline{z} = -4 - 2i$ **at** $(-4, -2)$ **(reflection in the real axis);** $-z = 4 - 2i$ **at** $(4, -2)$ **(rotation of** 180° **about** O**);** $iz = -4i + 2i^2 = -2 - 4i$ **at** $(-2, -4)$ **(rotation of** 90° **anticlockwise about** O**).**

2. Let $z = 2 + i$. Plot z, iz, i^2z and i^3z and name the shape they form.

> $z = 2 + i$, $iz = -1 + 2i$, $i^2z = -2 - i$, $i^3z = 1 - 2i$**. They are the vertices of a square centred at the origin.**

3. Show that if $z = a + ib$ then z, iz and O form a right-angled isosceles triangle.

> $iz = -b + ia$. $|z| = \sqrt{a^2 + b^2} = |iz|$**, so two sides are equal. The gradients of** Oz **and** $O(iz)$ **are** $\frac{b}{a}$ **and** $-\frac{a}{b}$**, whose product is** -1**, so the angle at** O **is a right angle.**

Lines, regions and loci

4. Sketch each region.

(a) $\operatorname{Re}(z) > 1$ (b) $-1 \le \operatorname{Im}(z) < 2$

(c) $\operatorname{Re}(z) \le 3$ and $\operatorname{Im}(z) \ge 0$

> **(a) The region to the right of the dashed line $x = 1$. (b) The horizontal strip between the solid line $y = -1$ and the dashed line $y = 2$. (c) The region to the left of the solid line $x = 3$ and on or above the real axis (both boundaries solid).**

5. Find and sketch the locus of z if (a) $z\overline{z} = 9$, (b) $(z+1)(\overline{z}+1) = 4$.

> **(a) $x^2 + y^2 = 9$: the circle with centre O and radius 3. (b) $(x+1)^2 + y^2 = 4$: the circle with centre $(-1, 0)$ and radius 2.**

6. Show that $z + \overline{z} = 2$ is a line, and find its Cartesian equation.

> **$z + \overline{z} = 2x = 2$, so $x = 1$: the vertical line $\operatorname{Re}(z) = 1$.**

HSC-style

7. Given that $\dfrac{1}{z} + \dfrac{1}{\overline{z}} = 2$, find the locus of z and sketch it, clearly indicating any excluded points.

> **$z \neq 0$. Multiplying by $z\overline{z}$: $\overline{z} + z = 2z\overline{z}$, so $2x = 2(x^2 + y^2)$, that is $x^2 - x + y^2 = 0$ and $\left(x - \frac{1}{2}\right)^2 + y^2 = \frac{1}{4}$. This is the circle with centre $\left(\frac{1}{2}, 0\right)$ and radius $\frac{1}{2}$, with the origin excluded (open circle at O).**