MCINTYRE MATHS · MATHEMATICS EXTENSION 2

Worked solutions

Lesson 3 · Quadratic equations and square roots

Warm-up · Retrieval from Lessons 1 and 2

(a) Simplify i^{103}.

> $-i$
> $103 = 4 \times 25 + 3$, so $i^{103} = i^3 = -i$.

(b) Expand $(3 - 2i)^2$.

> $5 - 12i$
> $9 - 12i + 4i^2 = 9 - 12i - 4 = 5 - 12i$. This expansion step is used for square roots later in the lesson.

(c) Express $\dfrac{2 + i}{1 - i}$ in the form $a + ib$.

> $\dfrac{1 + 3i}{2}$
> Multiply by the conjugate of the denominator: $\dfrac{(2 + i)(1 + i)}{(1 - i)(1 + i)} = \dfrac{2 + 2i + i - 1}{2} = \dfrac{1 + 3i}{2}$.

(d) Solve $x^2 + 16 = 0$.

> $x = \pm 4i$
> $x^2 = -16 = 16i^2$, so $x = \pm 4i$.

Example 1 · Quadratic formula

(a) Find the sum and the product of the roots.

> **Sum $= -2$, product $= 4$.**
> For $x^2 + 2x + 4 = 0$, the sum of the roots is $-\frac{b}{a} = -2$ and the product is $\frac{c}{a} = 4$.

(b) Use the quadratic formula to solve for x.

> $x = -1 \pm i\sqrt{3}$
> $x = \frac{-2 \pm \sqrt{4 - 16}}{2} = \frac{-2 \pm \sqrt{-12}}{2} = \frac{-2 \pm 2\sqrt{3}\,i}{2} = -1 \pm \sqrt{3}\,i$. Check: the sum is -2 and the product is $1 + 3 = 4$. Notice that the two roots are conjugates of each other.

Example 2 · Completing the square

Solve $z^2 + 8z + 25 = 0$ by completing the square.

> $z = -4 \pm 3i$
> $(z+4)^2 + 9 = 0$, so $(z+4)^2 = -9$, $z + 4 = \pm 3i$ and $z = -4 \pm 3i$. Note that a sum of squares factorises over $\mathbb{C}$: $z^2 + \lambda^2 = (z + i\lambda)(z - i\lambda)$.

Example 3

Find a quadratic equation with real coefficients that has $w = 2 - i$ as one root.

> $z^2 - 4z + 5 = 0$
> The other root is $\overline{w} = 2 + i$. Sum $= 4$, product $= 4 + 1 = 5$. Equivalently $(z - (2 - i))(z - (2 + i)) = (z - 2)^2 + 1 = 0$.

HSC-style · Coefficients from a root

The equation $z^2 + pz + q = 0$, where p and q are real, has $3 + i$ as a root. Find p and q.

> $p = -6$ **and** $q = 10$.
> The other root is $3 - i$. Sum of roots $= 6 = -p$, so $p = -6$. Product $= (3 + i)(3 - i) = 10 = q$. A common slip is forgetting the minus sign in $-p$ and giving $p = 6$.

Example 4

Find the square roots of -9.

> $\pm 3i$
> $(\pm 3i)^2 = 9i^2 = -9$. With the method: $x^2 - y^2 = -9$ and $2xy = 0$ give $x = 0$, $y = \pm 3$.

Example 5 · Square roots

Find the square roots of $7 + 24i$.

> $\pm(4 + 3i)$
> Let $(x + iy)^2 = 7 + 24i$. Then $x^2 - y^2 = 7$ (1) and $2xy = 24$, so $y = \frac{12}{x}$ (2). Substitute (2) into (1): $x^2 - \frac{144}{x^2} = 7$, so $x^4 - 7x^2 - 144 = 0$, that is $(x^2 - 16)(x^2 + 9) = 0$. Since x is real, $x^2 = 16$, so $x = \pm 4$ and $y = \pm 3$ (same sign). Check: $(4 + 3i)^2 = 16 - 9 + 24i$.

Example 6

(a) $-5 + 12i$

> $\pm(2 + 3i)$
> $x^2 - y^2 = -5$ and $x^2 + y^2 = 13$, so $x^2 = 4$, $y^2 = 9$. Since $b = 12 > 0$ the signs match: $\pm(2 + 3i)$.

Example 6 · A faster way

(b) $-i$

> $\pm\dfrac{1}{\sqrt{2}}(1-i)$
> $x^2-y^2=0$ and $x^2+y^2=1$, so $x^2=y^2=\frac{1}{2}$. Since $b=-1<0$ the signs are opposite: $\pm\frac{1}{\sqrt{2}}(1-i)$. Check: $(1-i)^2=-2i$, and $\frac{1}{2}(-2i)=-i$.

HSC-style · Complex coefficients

Solve $z^2+3z+(3-i)=0$.

> $z=-1+i$ **or** $z=-2-i$.
> $\Delta=9-4(3-i)=-3+4i$. For the square roots: $x^2-y^2=-3$ and $x^2+y^2=5$ give $x^2=1$, $y^2=4$, same sign, so $\pm(1+2i)$. Then $z=\frac{-3\pm(1+2i)}{2}$. Check: the sum is -3 and the product is $(-1+i)(-2-i)=2+i-2i+1=3-i$.

Example 7 · A harder one

Solve $z^2+2\left(1-i\sqrt{3}\right)z+2+2i\sqrt{3}=0$.

> $z=(-1+\sqrt{2})+(\sqrt{3}-\sqrt{6})\,i$ **or** $z=(-1-\sqrt{2})+(\sqrt{3}+\sqrt{6})\,i$.
> $\Delta=4(1-i\sqrt{3})^2-4(2+2i\sqrt{3})=4(-2-2\sqrt{3}\,i)-8-8\sqrt{3}\,i=-16-16\sqrt{3}\,i=16(-1-\sqrt{3}\,i)$. Square roots of $-1-\sqrt{3}\,i$: $x^2-y^2=-1$ and $x^2+y^2=2$ give $x^2=\frac{1}{2}$, $y^2=\frac{3}{2}$; $b<0$ so the signs are opposite: $\pm\left(\frac{1}{\sqrt{2}}-\sqrt{\frac{3}{2}}\,i\right)$. So $\sqrt{\Delta}=\pm(2\sqrt{2}-2\sqrt{6}\,i)$ and $z=-1+\sqrt{3}\,i\pm(\sqrt{2}-\sqrt{6}\,i)$. Never leave a complex number under a square root sign: find its square roots first.

HSC-style · Using square roots

(i) Find the square roots of $-i$. (ii) Hence solve $z^2+2z+(1+i)=0$.

> **(i)** $\pm\dfrac{1}{\sqrt{2}}(1-i)$**. (ii)** $z=-1+\dfrac{1}{\sqrt{2}}-\dfrac{1}{\sqrt{2}}\,i$ **or** $z=-1-\dfrac{1}{\sqrt{2}}+\dfrac{1}{\sqrt{2}}\,i$.
> (i) is Example 6(b). (ii) $\Delta=4-4(1+i)=-4i$, so $\sqrt{\Delta}=\pm2\cdot\frac{1}{\sqrt{2}}(1-i)=\pm\sqrt{2}(1-i)$ and $z=\frac{-2\pm\sqrt{2}(1-i)}{2}=-1\pm\frac{1}{\sqrt{2}}(1-i)$. Completing the square gives the same: $(z+1)^2=-i$.

Exit ticket

1. Find the square roots of $-3-4i$.

> $\pm(1-2i)$
> $x^2-y^2=-3$ and $x^2+y^2=5$, so $x^2=1$, $y^2=4$. Since $b=-4<0$ the signs are opposite. Check: $(1-2i)^2=1-4i-4=-3-4i$.

2. Hence solve $z^2-3z+(3+i)=0$.

> $z=2-i$ **or** $z=1+i$.
> $\Delta=9-4(3+i)=-3-4i$, so $\sqrt{\Delta}=\pm(1-2i)$ and $z=\frac{3\pm(1-2i)}{2}$. Check: the sum is 3 and the product is $(2-i)(1+i)=3+i$.

Practice worksheet

Real coefficients

1. Solve.

(a) $z^2 + 6z + 13 = 0$ (b) $2z^2 - 2z + 5 = 0$

> **(a)** $(z+3)^2 = -4$, **so** $z = -3 \pm 2i$. **(b)** $z = \frac{2 \pm \sqrt{4-40}}{4} = \frac{2 \pm 6i}{4} = \frac{1}{2} \pm \frac{3}{2}i$.

2. Find a quadratic equation with real coefficients that has $-1 + 3i$ as a root.

> **The other root is** $-1 - 3i$. **Sum** $= -2$, **product** $= 1 + 9 = 10$. **So** $z^2 + 2z + 10 = 0$.

3. The equation $z^2 + pz + q = 0$, where $p, q \in \mathbb{R}$, has $1 - 2i$ as a root. Find p and q.

> **The other root is** $1 + 2i$. **Sum** $= 2 = -p$, **so** $p = -2$. **Product** $= 1 + 4 = 5 = q$.

Square roots

4. Find the square roots of (a) -16, (b) $5 - 12i$, (c) $8 + 6i$.

> **(a)** $\pm 4i$. **(b)** $x^2 - y^2 = 5$, $x^2 + y^2 = 13$, **so** $x^2 = 9$, $y^2 = 4$, **opposite signs:** $\pm(3 - 2i)$. **(c)** $x^2 - y^2 = 8$, $x^2 + y^2 = 10$, **so** $x^2 = 9$, $y^2 = 1$, **same signs:** $\pm(3 + i)$.

5. Find the square roots of $1 + i\sqrt{3}$ in exact form.

> $x^2 - y^2 = 1$ **and** $x^2 + y^2 = 2$, **so** $x^2 = \frac{3}{2}$ **and** $y^2 = \frac{1}{2}$. **Since** $b > 0$ **the signs match:** $\pm\left(\sqrt{\frac{3}{2}} + \frac{1}{\sqrt{2}}\,i\right) = \pm\dfrac{\sqrt{6} + \sqrt{2}\,i}{2}$.

Complex coefficients

6. Solve $z^2 - (5+i)z + (8+i) = 0$.

> $\Delta = (5+i)^2 - 4(8+i) = 24 + 10i - 32 - 4i = -8 + 6i$. **Square roots:** $x^2 - y^2 = -8$, $x^2 + y^2 = 10$, **so** $x^2 = 1$, $y^2 = 9$, **same signs:** $\pm(1 + 3i)$. **Then** $z = \frac{5+i \pm (1+3i)}{2}$, **so** $z = 3 + 2i$ **or** $z = 2 - i$.

HSC-style

7. (a) Find the square roots of $5 + 12i$. (b) Hence solve $z^2 - 2z - 4 - 12i = 0$.

> **(a)** $x^2 - y^2 = 5$ **and** $x^2 + y^2 = 13$, **so** $x^2 = 9$, $y^2 = 4$, **same signs:** $\pm(3 + 2i)$. **(b)** $(z-1)^2 = 5 + 12i$, **so** $z - 1 = \pm(3 + 2i)$ **and** $z = 4 + 2i$ **or** $z = -2 - 2i$. **Check: the sum is** 2 **and the product is** $(4 + 2i)(-2 - 2i) = -4 - 12i$.