MCINTYRE MATHS · MATHEMATICS EXTENSION 2

Worked solutions

Lesson 2 · Conjugates, division and equality

Warm-up · Real products

(a) $(5+i)(5-i)$

> 26
> $25 - i^2 = 25 + 1 = 26$. The middle terms cancel.

(b) $(a+ib)(a-ib)$

> $a^2 + b^2$
> The same pattern as (a): $a^2 - i^2b^2 = a^2 + b^2$, which is always real.

(c) Rationalise the denominator of $\dfrac{1}{3-\sqrt{2}}$.

> $\dfrac{3+\sqrt{2}}{7}$
> Multiply top and bottom by $3+\sqrt{2}$: $\frac{3+\sqrt{2}}{9-2}$. This is the template for dividing complex numbers.

Example 1

(a) $\overline{-1+4i}$

> $-1-4i$
> Change the sign of the imaginary part.

(b) $\overline{-2i}$

> $2i$
> $-2i = 0 - 2i$, so its conjugate is $0 + 2i$.

(c) If $z = 3+4i$, find $z\overline{z}$ and $|z|$.

> $z\overline{z} = 25$ **and** $|z| = 5$.
> $(3+4i)(3-4i) = 9 + 16 = 25$, and $|z| = \sqrt{25} = 5$.

(d) $\overline{\overline{2+3i}}$

> $2+3i$
> Conjugating twice returns the original number.

Example 2 · Proving the conjugate rules

(a) Prove that $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$.

$$z_1 + z_2 = (a + c) + i(b + d)$$
$$\overline{z_1 + z_2} = (a + c) - i(b + d) = (a - ib) + (c - id) = \overline{z_1} + \overline{z_2}$$

Add first, then conjugate; the result matches conjugating first, then adding.

(b) Prove that $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$.

$$z_1 z_2 = (ac - bd) + i(ad + bc)$$
$$\overline{z_1 z_2} = (ac - bd) - i(ad + bc)$$
$$\overline{z_1}\,\overline{z_2} = (a - ib)(c - id) = ac - iad - ibc + i^2 bd = (ac - bd) - i(ad + bc)$$

The two sides are equal.

Expand both sides and compare real and imaginary parts. These identities are part of the syllabus: you must be able to prove and use them.

Example 3 · Using the conjugate rules

(a) $\overline{(5 + i) + (3 - 2i)}$

$8 + i$

Directly: $(8 - i)$ conjugated is $8 + i$. By the rule: $(5 - i) + (3 + 2i) = 8 + i$.

(b) $\overline{(5 + i)(3 - 2i)}$

$17 + 7i$

By the rule: $(5 - i)(3 + 2i) = 15 + 10i - 3i + 2 = 17 + 7i$. Directly: $(5 + i)(3 - 2i) = 17 - 7i$, whose conjugate is $17 + 7i$.

Example 4

(a) $\dfrac{3}{2i}$

$-\dfrac{3}{2}\,i$

Multiply by $\frac{i}{i}$: $\frac{3i}{2i^2} = -\frac{3i}{2}$. Multiplying by $\frac{-2i}{-2i}$ works too, but is slower.

(b) $\dfrac{1}{3 + 2i}$

$\dfrac{3}{13} - \dfrac{2}{13}\,i$

$\frac{1}{3+2i} \times \frac{3-2i}{3-2i} = \frac{3-2i}{9+4}$.

Example 5 · Division

Express $\dfrac{3+4i}{2-3i}$ in the form $a+ib$.

$$-\frac{6}{13}+\frac{17}{13}\,i$$

$$\frac{(3+4i)(2+3i)}{(2-3i)(2+3i)}=\frac{6+9i+8i-12}{4+9}=\frac{-6+17i}{13}$$

Check the answer on a calculator in complex mode (the syllabus says “with and without digital tools”).

Example 6

(a) Prove each relationship.

Let $z=a+ib$. Then $z+\overline{z}=2a$, so $a=\frac{z+\overline{z}}{2}$. Also $z-\overline{z}=2ib$, so $b=\frac{z-\overline{z}}{2i}$. Finally $\frac{1}{z}=\frac{\overline{z}}{z\overline{z}}=\frac{\overline{z}}{|z|^2}$.
Add and subtract z and its conjugate, then use $z\overline{z}=|z|^2$.

(b) Hence show that z is real if and only if $z=\overline{z}$.

z is real $\Leftrightarrow \operatorname{Im}(z)=0 \Leftrightarrow \frac{z-\overline{z}}{2i}=0 \Leftrightarrow z=\overline{z}$.
Every step is reversible, so the argument proves both directions.

(c) What can you say about $z\neq 0$ if $\overline{z}=-z$?

z is purely imaginary.
$\overline{z}=-z \Leftrightarrow z+\overline{z}=0 \Leftrightarrow 2\operatorname{Re}(z)=0 \Leftrightarrow \operatorname{Re}(z)=0.$

Example 7

Find real a and b such that $z=a+ib$ satisfies $z+3i\overline{z}=-1+13i$.

$a=5, b=-2$, **so $z=5-2i$.**
$z+3i\overline{z}=(a+ib)+3i(a-ib)=(a+3b)+i(3a+b)$. Real parts: $a+3b=-1$. Imaginary parts: $3a+b=13$. Eliminating: $3\times(\text{first})-(\text{second})$ gives $8b=-16$, so $b=-2$ and $a=5$.
Check: $5-2i+3i(5+2i)=5-2i+15i-6=-1+13i$.

HSC practice · Conjugates and division

Let $z=5+i$ and $w=2-4i$. Express $\dfrac{\overline{z}}{w}$ in the form $x+iy$.

$$\frac{7}{10}+\frac{9}{10}\,i$$

$\frac{5-i}{2-4i}\times\frac{2+4i}{2+4i}=\frac{10+20i-2i+4}{20}=\frac{14+18i}{20}$.

HSC practice · Conjugates and division

Let $z = 1 + 3i$ and $w = 2 - i$. Find, in the form $x + iy$: (i) $\overline{z}\,w$; (ii) $\dfrac{z}{w}$.

(i) $-1 - 7i$. (ii) $-\dfrac{1}{5} + \dfrac{7}{5}\,i$.
(i) $(1 - 3i)(2 - i) = 2 - i - 6i + 3i^2 = -1 - 7i$. (ii) $\frac{(1+3i)(2+i)}{5} = \frac{2+i+6i-3}{5} = \frac{-1+7i}{5}$.

HSC practice · Equating parts

Let $z = 2 + 3i$ and $w = 1 - 5i$. Find in the form $x + iy$: (i) $z + \overline{w}$; (ii) z^2; (iii) $\dfrac{z}{w}$.

(i) $3 + 8i$. (ii) $-5 + 12i$. (iii) $-\dfrac{1}{2} + \dfrac{1}{2}\,i$.
(i) $\overline{w} = 1 + 5i$, so $z + \overline{w} = 3 + 8i$. (ii) $z^2 = 4 + 12i + 9i^2 = -5 + 12i$. (iii) $\frac{(2+3i)(1+5i)}{|w|^2} = \frac{2+10i+3i-15}{26} = \frac{-13+13i}{26}$.

HSC practice · Equating parts

Find the complex number z such that $|z| = z + 8 + 12i$.

$z = 5 - 12i$
$|z|$ is real, so the imaginary part of the right-hand side must be zero. Let $z = a + ib$: then $b + 12 = 0$, so $b = -12$. The real parts give $\sqrt{a^2 + 144} = a + 8$, so $a^2 + 144 = a^2 + 16a + 64$ and $a = 5$. Check: $a + 8 = 13 > 0$ and $|5 - 12i| = 13$.

Exit ticket

1. Express $\dfrac{5}{1 - 2i}$ in the form $a + ib$.

$1 + 2i$
$\frac{5(1+2i)}{(1-2i)(1+2i)} = \frac{5(1+2i)}{5} = 1 + 2i$.

2. Find real x and y such that $(x + iy)(2 - 3i) = -13i$.

$x = 3, y = -2$
Expanding: $(2x + 3y) + i(2y - 3x) = -13i$, so $2x + 3y = 0$ and $2y - 3x = -13$, giving $x = 3$, $y = -2$. Alternatively divide: $x + iy = \frac{-13i(2+3i)}{13} = -i(2 + 3i) = 3 - 2i$.

Practice worksheet

Conjugates and modulus

1. Find $\overline{z}$, $z\overline{z}$ and $|z|$ for (a) $z = 5 - 12i$, (b) $z = -\sqrt{3} + i$.

(a) $\overline{z} = 5 + 12i$, $z\overline{z} = 25 + 144 = 169$, $|z| = 13$. (b) $\overline{z} = -\sqrt{3} - i$, $z\overline{z} = 3 + 1 = 4$, $|z| = 2$.

2. Let $z_1 = 2 - i$ and $z_2 = 4 + 3i$. Verify that $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$.

$z_1 z_2 = 8 + 6i - 4i - 3i^2 = 11 + 2i$, **so** $\overline{z_1 z_2} = 11 - 2i$. **Also** $\overline{z_1}\,\overline{z_2} = (2 + i)(4 - 3i) = 8 - 6i + 4i + 3 = 11 - 2i$. **They agree.**

3. Prove that $\overline{z_1 - z_2} = \overline{z_1} - \overline{z_2}$.

> **Let $z_1 = a + ib$ and $z_2 = c + id$. Then $z_1 - z_2 = (a - c) + i(b - d)$, so $\overline{z_1 - z_2} = (a - c) - i(b - d) = (a - ib) - (c - id) = \overline{z_1} - \overline{z_2}$.**

Division

4. Express in the form $a + ib$.

(a) $\dfrac{4}{i}$ (b) $\dfrac{2}{1+i}$

(c) $\dfrac{1+2i}{3-i}$ (d) $\dfrac{5+i}{2+3i}$

> **(a)** $\frac{4i}{i^2} = -4i$. **(b)** $\frac{2(1-i)}{2} = 1 - i$. **(c)** $\frac{(1+2i)(3+i)}{10} = \frac{3+i+6i-2}{10} = \frac{1}{10} + \frac{7}{10}i$. **(d)** $\frac{(5+i)(2-3i)}{13} = \frac{10-15i+2i+3}{13} = 1 - i$.

5. Use $\frac{1}{z} = \frac{\overline{z}}{|z|^2}$ to find $\frac{1}{z}$ when $z = -3 + 4i$.

> **$|z|^2 = 25$ and $\overline{z} = -3 - 4i$, so $\frac{1}{z} = \frac{-3-4i}{25} = -\frac{3}{25} - \frac{4}{25}i$.**

Equality of complex numbers

6. Find real x and y such that $(x + iy)(1 + 2i) = 7 + 4i$.

> **$(x - 2y) + i(2x + y) = 7 + 4i$, so $x - 2y = 7$ and $2x + y = 4$. Solving: $x = 3$, $y = -2$.**

7. Show that $z = \overline{z}$ if and only if z is real.

> **Let $z = a + ib$. Then $z = \overline{z} \Leftrightarrow a + ib = a - ib \Leftrightarrow 2ib = 0 \Leftrightarrow b = 0 \Leftrightarrow z$ is real.**

Adapted HSC-style

8. Let $z = 3 - 2i$ and $w = 1 + 4i$. (a) Express $\dfrac{w}{z}$ in the form $x + iy$. (b) Find real a and b such that $z + (a + ib)\overline{w} = 10 + i$.

> **(a)** $\frac{(1+4i)(3+2i)}{13} = \frac{3+2i+12i-8}{13} = -\frac{5}{13} + \frac{14}{13}i$. **(b)** $(a + ib)(1 - 4i) = (a + 4b) + i(b - 4a)$, **so** $3 + a + 4b = 10$ **and** $-2 + b - 4a = 1$, **that is** $a + 4b = 7$ **and** $b - 4a = 3$. **Then** $b = 4a + 3$ **gives** $17a + 12 = 7$, **so** $a = -\frac{5}{17}$, $b = \frac{31}{17}$.