MCINTYRE MATHS · MATHEMATICS EXTENSION 2

Worked solutions

Lesson 10 · Rays and regions

Retrieval from Lessons 5 and 9

(a) Find $\operatorname{Arg}(-1-i)$.

> $-\dfrac{3\pi}{4}$
> $-1-i$ is in the third quadrant with reference angle $\frac{\pi}{4}$. The principal argument lies in $(-\pi, \pi]$, so it is $-\pi + \frac{\pi}{4} = -\frac{3\pi}{4}$.

(b) Describe the region $|z-1| \leq 2$.

> **The closed disc with centre 1 and radius 2 (solid boundary).**
> $|z-1|$ is the distance from 1, and "$\leq$" includes the circle itself.

(c) A circle has centre $(2, 0)$ and radius 1. A tangent from O touches it at T. Find $\angle TOX$, where X is on the positive real axis.

> $\dfrac{\pi}{6}$
> The radius to T is perpendicular to the tangent, so in the right-angled triangle $\sin \angle TOX = \frac{1}{2}$ and $\angle TOX = \frac{\pi}{6}$. This is used again in Example 4.

(d) Write $\tan \dfrac{\pi}{3}$ exactly.

> $\sqrt{3}$
> From the $30°$-$60°$-$90°$ triangle, $\tan 60° = \frac{\sqrt{3}}{1}$.

Example 1 · Rays

(a) $\arg z = \dfrac{\pi}{6}$

> **A ray from O at angle $\frac{\pi}{6}$ to the positive real axis, with an open circle at O.**
> $\arg z = \frac{\pi}{6}$ means $z = r\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$ with $r > 0$. O is excluded.

(b) $\arg\left(z-(1+i)\right) = \dfrac{\pi}{6}$

> **A ray from $(1, 1)$ at angle $\frac{\pi}{6}$ (parallel to the ray in (a)), with an open circle at $1+i$.**
> Common errors: drawing a full line, or starting the ray at O instead of at $1+i$.

Example 2 · More rays

(a) $\arg(z+2) = \dfrac{3\pi}{4}$

> **A ray from -2 going up and to the left at $135°$, with an open circle at -2.**
> $z+2 = z-(-2)$, so the ray starts at -2.

(b) $\arg(z-1+i) = \arg(z+1-i)$

> **Two rays on the line $y=-x$: one starting at $(1,-1)$ going down-right, and one starting at $(-1,1)$ going up-left. Both endpoints are excluded, and the segment between them is not part of the locus.**
> The directions from $(1,-1)$ and from $(-1,1)$ to Z are equal. This happens only on the line through the two points, and only when Z is outside the interval between them (inside it the two directions are opposite).

Example 3 · Regions

(a) $1 \le |z| \le 3$

> **The annulus between the circles of radius 1 and 3 about O, both boundaries solid.**
> $|z| \ge 1$ is outside or on the unit circle and $|z| \le 3$ is inside or on the circle of radius 3.

(b) $\dfrac{\pi}{6} \le \arg z \le \dfrac{\pi}{3}$

> **The sector between the rays at $\frac{\pi}{6}$ and $\frac{\pi}{3}$ from O, rays solid, with an open circle at O.**
> O is excluded because $\arg 0$ is undefined.

HSC practice · Regions

Sketch the region where $\mathrm{Re}(z) > 1$ and $\mathrm{Arg}\, z < \dfrac{\pi}{3}$.

> **The region to the right of the dashed line $x=1$ and below the dashed ray $y=\sqrt{3}\,x$ (the ray from O at $\frac{\pi}{3}$). The corner $(1,\sqrt{3})$ is not included.**
> Points with $\mathrm{Re}(z) > 1$ already have $\mathrm{Arg}\, z > -\frac{\pi}{2}$, so only the upper bound matters. The two boundaries meet at $(1,\sqrt{3})$.

HSC practice · Regions

Sketch the region where $|z| < 3$ and $0 \le \arg(z-i) \le \dfrac{\pi}{2}$.

> **Inside the dashed circle of radius 3 about O, in the quarter-plane $x \ge 0$, $y \ge 1$ with corner at i. The two straight edges (from i) are solid, and the point i itself is excluded (open circle).**
> $0 \le \arg(z-i) \le \frac{\pi}{2}$ says that $z-i$ lies in the first quadrant, including its two axes but not the origin. The edges end where they meet the circle, at $\left(\sqrt{8},1\right)$ and $(0,3)$. Marks are usually for the dashed circle, the quarter-plane from i, the open point at i and the correct shading.

Example 4 · Greatest and least values

P represents z, where $|z-2|=1$. (i) Sketch the locus. (ii) Find the greatest and least values of $|z|$ and of $\arg z$. (iii) OP_1 and OP_2 are tangents to the locus. Write z_1 and z_2 in polar form.

(i) Circle with centre 2 and radius 1. (ii) $1 \le |z| \le 3$ and $-\frac{\pi}{6} \le \arg z \le \frac{\pi}{6}$. (iii) $z_1 = \sqrt{3}\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$ and $z_2 = \sqrt{3}\left(\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right)$.
(ii) Along the real axis the nearest and farthest points are 1 and 3. A tangent from O is perpendicular to the radius, so in the right-angled triangle $\sin\alpha = \frac{1}{2}$ and $\alpha = \frac{\pi}{6}$. (iii) The tangent length is $\sqrt{2^2 - 1^2} = \sqrt{3}$. As a check, $z_1^2 + z_2^2 = 3\,\mathrm{cis}\,\frac{\pi}{3} + 3\,\mathrm{cis}\left(-\frac{\pi}{3}\right) = 3$.

Example 5

Sketch the region where $|z-1-i| \le \sqrt{2}$ and $0 \le \arg z \le \dfrac{\pi}{4}$. Find the greatest value of $|z|$ in the region and the point where it occurs.

The greatest value is $|z| = 2\sqrt{2}$, at $z = 2+2i$.
The circle with centre $(1,1)$ and radius $\sqrt{2}$ passes through O and meets the real axis at O and 2. The diagonal $y = x$ passes through the centre, so the region is the half-disc below $y = x$ minus the small segment below the real axis. It is bounded by the chord from O to 2 on the real axis, the arc from 2 to $2+2i$, and the diameter from $2+2i$ back to O. O is excluded because $\arg 0$ is undefined. The farthest point from O is the other end of the diameter through O, $2+2i$, which lies on $\arg z = \frac{\pi}{4}$ (included).

Example 6

Sketch the locus given by $\arg(z-2) - \arg(z+2) = \dfrac{\pi}{3}$, and find the centre and radius of the circle.

The major arc, above the real axis, of the circle with centre $\left(0, \dfrac{2}{\sqrt{3}}\right)$ and radius $\dfrac{4}{\sqrt{3}}$, with ± 2 excluded.
$\arg(z-2) - \arg(z+2)$ is the angle at Z in the triangle with vertices -2, 2 and Z. The chord of length 4 subtends $\frac{\pi}{3}$ at the circumference, so $2R\sin\frac{\pi}{3} = 4$ and $R = \frac{4}{\sqrt{3}}$. The centre is on the imaginary axis at height $R\cos\frac{\pi}{3} = \frac{2}{\sqrt{3}}$. The circle is $x^2 + \left(y - \frac{2}{\sqrt{3}}\right)^2 = \frac{16}{3}$ with $y > 0$.
(Algebraically, $4y = \sqrt{3}(x^2 + y^2 - 4)$ gives the same circle.)

Spot the error · Rays and regions

(a) $\arg z = \dfrac{\pi}{4}$ is the line $y = x$.

It is only the ray $y = x$, $x > 0$, with O excluded.
A fixed argument gives a ray, not a full line. $\arg 0$ is undefined, so O is an open circle.

(b) $\arg(z-1) = \dfrac{\pi}{2}$ is a ray starting at O.

The ray starts at 1: it is the vertical ray $x = 1$, $y > 0$, with the point 1 excluded.
$z - 1 = z - (1)$, so the ray starts at 1, not at O.

(c) In $1 \leq |z| \leq 3$ both circles are dashed.

> **Both circles are solid.**
> "$\leq$" includes the boundary, so both circles are part of the region.

(d) The region $\dfrac{\pi}{6} \leq \arg z \leq \dfrac{\pi}{3}$ includes the origin.

> **The origin is excluded (open circle).**
> $\arg 0$ is undefined, so O cannot satisfy the condition.

Exit ticket

1. Sketch the region $|z-2| \leq 2$ and $\operatorname{Im}(z) \geq 1$.

> **The part of the closed disc with centre 2 and radius 2 on or above the solid line $y = 1$ (a segment of the disc).**
> The line $y = 1$ cuts the circle where $(x-2)^2 + 1 = 4$, that is $x = 2 \pm \sqrt{3}$.

2. Find the least value of $|z|$ if $\arg(z - 2i) = -\dfrac{\pi}{4}$.

> **$\sqrt{2}$, at $z = 1 + i$.**
> The locus is the ray from $2i$ with points $(s, 2-s)$, $s > 0$. Then $|z|^2 = s^2 + (2-s)^2 = 2(s-1)^2 + 2$ is least at $s = 1$. (This is the foot of the perpendicular from O to the line, and it lies on the ray.)

Practice worksheet

Rays

1. Sketch each locus and describe it in words.

(a) $\arg(z-2) = \frac{\pi}{2}$ (b) $\arg(z+1-i) = -\frac{\pi}{3}$

(c) $\arg(z-i) = \arg(z+i)$

> **(a) The vertical ray from 2 upwards ($x = 2$, $y > 0$), 2 excluded. (b) The ray from $-1+i$ pointing down-right at $60°$ below the horizontal, with $-1+i$ excluded. (c) The two rays on the imaginary axis above i and below $-i$ (directions from i and $-i$ to Z are equal only outside the interval between them); both endpoints excluded.**

Regions

2. Sketch each region.

(a) $|z| \leq 2$ and $\operatorname{Re}(z) \geq 0$ (b) $2 < |z-1| < 3$

(c) $-\frac{\pi}{4} < \arg z \leq \frac{\pi}{4}$

> **(a) The right half of the closed disc of radius 2, boundaries solid. (b) The annulus between the circles of radius 2 and 3 about 1, both dashed. (c) The sector between the rays at $-\frac{\pi}{4}$ (dashed) and $\frac{\pi}{4}$ (solid), O excluded.**

3. Sketch the region where $|z - i| \le 2$ and $\operatorname{Im}(z) \ge 0$, and find its area. (A segment of a circle of radius r with chord subtending angle α has area $\frac{1}{2}r^2(\alpha - \sin\alpha)$.)

> **The line $y = 0$ is 1 below the centre, so it cuts the circle at $x = \pm\sqrt{3}$, and the chord subtends $\alpha = \frac{2\pi}{3}$ at the centre. The part below $y = 0$ is a segment of area $\frac{1}{2} \cdot 4\left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right) = \frac{4\pi}{3} - \sqrt{3}$. The region is the disc minus this segment: $4\pi - \frac{4\pi}{3} + \sqrt{3} = \frac{8\pi}{3} + \sqrt{3}$.**

Greatest and least values

4. $|z - 3 - 4i| = 2$. Find the greatest and least values of $|z|$.

> **The centre is $3 + 4i$ with $|3 + 4i| = 5$. Least $|z| = 5 - 2 = 3$ and greatest $|z| = 5 + 2 = 7$.**

5. The complex number z satisfies $|z - 4| = 2$. Find the greatest value of $\arg z$ and the corresponding z.

> **The tangent from O makes $\sin\alpha = \frac{2}{4}$, so $\alpha = \frac{\pi}{6}$. Tangent length $\sqrt{16 - 4} = 2\sqrt{3}$. $z = 2\sqrt{3}\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) = 3 + \sqrt{3}\,i$. Greatest $\arg z = \frac{\pi}{6}$.**

Arcs (extension)

6. Sketch $\arg\left(\dfrac{z-1}{z+1}\right) = \dfrac{\pi}{2}$. Describe the locus.

> **The angle subtended at Z by the interval from -1 to 1 is a right angle, so Z lies on the semicircle above the real axis with diameter $[-1, 1]$: $x^2 + y^2 = 1$, $y > 0$, with ± 1 excluded.**

HSC-style

7. (a) Sketch the region where $|z - 2| \le 2$ and $0 \le \arg z \le \dfrac{\pi}{4}$. (b) Find the greatest value of $|z|$ in the region and the complex number where it occurs.

> **(a) The circle has centre 2 and radius 2 and passes through O. The region lies between the real axis and the line $y = x$ inside the disc, with O excluded. (b) The point of the disc farthest from O is the far end of the diameter through O, which is $z = 4$. It lies on the real axis, so $\arg z = 0$ is included in the region, and the greatest value of $|z|$ is 4. (The line $y = x$ meets the circle at $2 + 2i$, where $|z| = 2\sqrt{2} < 4$.)**