MCINTYRE MATHS · MATHEMATICS EXTENSION 2

Worked solutions

Lesson 1 · The arithmetic of complex numbers

Warm-up · Why do we need a new number?

(a) Find the sum and the product of the roots.

> **Sum $= -2$, product $= 4$.**
> For $ax^2 + bx + c = 0$ the sum of the roots is $-\frac{b}{a}$ and the product is $\frac{c}{a}$. These formulae still make sense even if the roots are not real.

(b) Use the quadratic formula. What goes wrong?

> $x = \dfrac{-2 \pm \sqrt{4 - 16}}{2} = \dfrac{-2 \pm \sqrt{-12}}{2}$**. The discriminant is $\Delta = -12 < 0$, so there are no real roots.**
> We need a number whose square is negative. Once we have one, the roots are $x = -1 \pm i\sqrt{3}$.
> Check: the sum is -2 and the product is $1 + 3 = 4$.

Discuss

If multiplying by a number i twice has the same effect as multiplying by -1, what must i^2 equal? What does multiplying by i do to the number line?

> **$i^2 = -1$. Multiplying by i is a quarter-turn ($90°$): it takes 1 off the real number line to a new vertical axis.**
> This is only a preview. In Lesson 4 we plot complex numbers on the Argand diagram, and in Lesson 8 we use multiplication by i as a rotation.

Example 1 · Solve

(a) $x^2 + 9 = 0$

> $x = \pm 3i$
> $x^2 = -9 = 9 \times (-1)$, so $x = \pm 3i$.

(b) $x^2 + 5 = 0$

> $x = \pm i\sqrt{5}$
> $x^2 = -5$, so $x = \pm\sqrt{5}\,i$.

(c) $4x^2 + 49 = 0$

> $x = \pm\dfrac{7i}{2}$
> $x^2 = -\frac{49}{4}$, so $x = \pm\frac{7}{2}i$.

Example 2 · Powers of i

(a) i^2

> -1
> This is the definition of i.

(b) i^3

> $-i$
> $i^3 = i^2 \times i = -i$.

(c) i^4

> 1
> $i^4 = (i^2)^2 = (-1)^2 = 1$.

(d) i^5

> i
> $i^5 = i^4 \times i = i$.

(e) i^6

> -1
> $i^6 = i^4 \times i^2 = -1$.

(f) i^7

> $-i$
> $i^7 = i^4 \times i^3 = -i$. Can you spot the cycle of four before the next slide?

(g) i^8

> 1
> $i^8 = (i^4)^2 = 1$.

(h) i^{40}

> 1
> $i^{40} = (i^4)^{10} = 1^{10} = 1$.

(i) i^{50}

> -1
> $i^{50} = (i^4)^{12} \times i^2 = -1$.

(j) i^{231}

> $-i$
> $231 = 4 \times 57 + 3$, so $i^{231} = (i^4)^{57} \times i^3 = -i$.

HSC question

Evaluate $i + i^2 + i^3 + i^4 + i^5$.

> i
> $i + (-1) + (-i) + 1 + i = i$. Any four consecutive powers of i add to 0, so only the fifth term survives. Extension: $i + i^2 + \cdots + i^{2027} = -1$, since $2027 = 4 \times 506 + 3$ leaves $i + i^2 + i^3 = -1$.

Example 3 · Real and imaginary parts

(a) $3 + 4i$

> $\mathrm{Re} = 3,\ \mathrm{Im} = 4$
> $\mathrm{Im}(z)$ is 4, not $4i$.

(b) $-2i$

> $\mathrm{Re} = 0,\ \mathrm{Im} = -2$
> $-2i = 0 + (-2)i$, so z is purely imaginary.

(c) 7

> $\mathrm{Re} = 7,\ \mathrm{Im} = 0$
> $7 = 7 + 0i$. Every real number is a complex number.

(d) $\dfrac{1 - i\sqrt{5}}{2}$

> $\mathrm{Re} = \dfrac{1}{2},\ \mathrm{Im} = -\dfrac{\sqrt{5}}{2}$
> Split it as $\frac{1}{2} - \frac{\sqrt{5}}{2}i$.

Example 4 · Classifying numbers

(a) $5,\quad \dfrac{18}{6}$

> $5 \in \mathbb{N}$ **and** $\frac{18}{6} = 3 \in \mathbb{N}$
> Whole numbers 1, 2, 3, … (some texts include 0 in $\mathbb{N}$; the syllabus does not insist either way).

(b) -8

> $\mathbb{Z}$
> A negative whole number is an integer but not a natural number.

(c) $\sin\dfrac{\pi}{6},\quad \dfrac{22}{7}$

> $\sin\frac{\pi}{6} = \frac{1}{2} \in \mathbb{Q}$ **and** $\frac{22}{7} \in \mathbb{Q}$
> Both are ratios of integers. Note $\frac{22}{7}$ is only an approximation of π.

(d) $\pi,\quad \sqrt{5}$

> $\mathbb{R}$ **(irrational)**
> They are real but cannot be written as a ratio of integers.

(e) $\sqrt{49}$

> $7 \in \mathbb{N}$
> $\sqrt{49} = 7$ exactly, so it is a natural number.

(f) $2 + 3i, \quad 4i$

> $\mathbb{C}$
> Both are non-real, so $\mathbb{C}$ is the smallest set containing them.

Example 5

(a) $(3 + 4i) + (2 + 6i)$

> $5 + 10i$
> Add real parts and add imaginary parts.

(b) $(4 - 3i) - (2 - 4i)$

> $2 + i$
> $(4 - 2) + (-3 + 4)i = 2 + i$. Take care with the signs when subtracting.

(c) $(3 + 2i)(4 + 2i)$

> $8 + 14i$
> $12 + 6i + 8i + 4i^2 = 12 + 14i - 4 = 8 + 14i$.

Example 6 · Multiplying

(a) $(2 + 5i)(3 + 6i)$

> $-24 + 27i$
> $6 + 12i + 15i + 30i^2 = 6 + 27i - 30 = -24 + 27i$.

(b) $(3 + 2i)(3 - 2i)$

> 13
> $9 - 6i + 6i - 4i^2 = 9 + 4 = 13$. The middle terms cancel.

(c) $(a + ib)(a - ib)$

> $a^2 + b^2$
> The product is always real. The numbers $a + ib$ and $a - ib$ are called **conjugates** (Lesson 2).

HSC practice · Expanding a power

Expand $(a + ib)^3$, where $a, b \in \mathbb{R}$, and write it in the form $x + iy$. Hence find $(2 - i)^3$.

> $(a + ib)^3 = (a^3 - 3ab^2) + i(3a^2b - b^3)$, **and** $(2 - i)^3 = 2 - 11i$.
> $(a + ib)^3 = a^3 + 3a^2(ib) + 3a(ib)^2 + (ib)^3 = a^3 + 3a^2b\,i - 3ab^2 - b^3i$, using $i^2 = -1$ and $i^3 = -i$. For $(2 - i)^3$ put $a = 2, b = -1$: real part $8 - 6 = 2$, imaginary part $-12 + 1 = -11$. A common slip is forgetting that $i^3 = -i$.

Exit ticket

1. Simplify i^{2027}.

> $-i$
> $2027 = 4 \times 506 + 3$, so $i^{2027} = (i^4)^{506} \times i^3 = -i$.

2. Express $(1+i)^4$ in the form $a + ib$.

> -4
> $(1+i)^2 = 1 + 2i + i^2 = 2i$, so $(1+i)^4 = (2i)^2 = 4i^2 = -4$.

3. Find $\text{Re}(z)$ and $\text{Im}(z)$ if $z = (2-i)(3+4i)$.

> $\text{Re}(z) = 10, \text{Im}(z) = 5$
> $z = 6 + 8i - 3i - 4i^2 = 6 + 5i + 4 = 10 + 5i$.

Practice worksheet

The number i and powers of i

1. Solve each equation.

(a) $x^2 + 16 = 0$ (b) $x^2 + 7 = 0$

(c) $9x^2 + 4 = 0$

> **(a)** $x = \pm 4i$. **(b)** $x = \pm i\sqrt{7}$. **(c)** $x^2 = -\frac{4}{9}$, **so** $x = \pm\frac{2i}{3}$.

2. Simplify.

(a) i^{12} (b) i^{27}

(c) i^{102} (d) i^{-1}

> **(a)** $i^{12} = (i^4)^3 = 1$. **(b)** $27 = 4 \times 6 + 3$, **so** $i^{27} = -i$. **(c)** $102 = 4 \times 25 + 2$, **so** $i^{102} = -1$. **(d)** $i^{-1} = \frac{1}{i} = \frac{i}{i^2} = -i$.

Number sets and parts

3. State the smallest set ($\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ or $\mathbb{C}$) containing each number.

(a) $-\frac{3}{4}$ (b) $\sqrt{16}$

(c) $\sqrt{12}$ (d) $5i - 2$

(e) $\cos\pi$

> **(a)** $\mathbb{Q}$. **(b)** $\sqrt{16} = 4$, **so** $\mathbb{N}$. **(c)** $\mathbb{R}$ **(irrational).** **(d)** $\mathbb{C}$. **(e)** $\cos\pi = -1$, **so** $\mathbb{Z}$.

4. Write down $\text{Re}(z)$ and $\text{Im}(z)$ for (a) $z = -5 + 2i$, (b) $z = 6i$, (c) $z = \frac{3-i}{4}$.

> **(a)** $\text{Re} = -5, \text{Im} = 2$. **(b)** $\text{Re} = 0, \text{Im} = 6$. **(c)** $z = \frac{3}{4} - \frac{1}{4}i$: $\text{Re} = \frac{3}{4}, \text{Im} = -\frac{1}{4}$.

Arithmetic

5. Express in the form $a + ib$.

(a) $(5 - 2i) + (1 + 7i)$ (b) $(2 + 3i) - (6 - i)$

(c) $(4 + i)(2 - 3i)$ (d) $(1 - 2i)^2$

> **(a)** $6 + 5i$. **(b)** $-4 + 4i$. **(c)** $8 - 12i + 2i - 3i^2 = 11 - 10i$. **(d)** $1 - 4i + 4i^2 = -3 - 4i$.

6. Find $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$ if $z = (2 - i)(3 + 4i)$.

> $z = 6 + 8i - 3i - 4i^2 = 10 + 5i$, **so** $\mathrm{Re}(z) = 10$ **and** $\mathrm{Im}(z) = 5$.

Adapted HSC-style

7. Let $z = 1 + i$. (a) Find z^2 and z^3 in the form $a + ib$. (b) Hence evaluate $z + z^2 + z^3$. (c) Find z^8.

> **(a)** $z^2 = 1 + 2i + i^2 = 2i$ **and** $z^3 = 2i(1 + i) = 2i - 2 = -2 + 2i$. **(b)** $(1 + i) + 2i + (-2 + 2i) = -1 + 5i$. **(c)** $z^8 = (z^2)^4 = (2i)^4 = 16i^4 = 16$.